Fundamental Solution of Static Equations of Transversely Isotropic Plates

Igor Bokov
A. N. Podgorny Institute for Mechanical Engineering Problems of the National Academy of Sciences of Ukraine, Kharkov, Ukraine,

Elena Strelnikova
A. N. Podgorny Institute for Mechanical Engineering Problems of the National Academy of Sciences of Ukraine, Kharkov, Ukraine,

ABSTRACT
A fundamental solution of the elasticity theory equations for transversely isotropic plates was obtained. To construct the two-dimensional elasticity theory equations, the approximation method of displacements, stresses and strains using Fourier series by Legendre polynomials on the transverse coordinate was used. This approach has allowed to take into account the transverse shear and normal stresses. Since the classical Kirchhoff-Love theory does not consider these stresses, the research based on the refined theories of the stress-strain state of transversely isotropic plates under concentrated force actions is an urgent scientific and technical problem. The fundamental solution of these equations was found using the two-dimensional Fourier integral transform and the generalization method, built with a special G-function. The method allows to reduce the system of resolvent differential static equations of flat plates and shells to a system of algebraic equations. Then the inverse Fourier transform restores fundamental solution. Numerical studies that demonstrate behavior patterns of the stress-strain state components depending on the elastic constants of transversely isotropic material were performed. The approach demonstrates the development of the refined theory of plates and shells based on the three-dimensional elasticity theory.

Keywords
Transversely isotropic plates, refined theory, force actions, special G-function

1. INTRODUCTION
Engineering structures from thin-walled structural components exposed to considerable force actions are widely used in modern technology. Additional difficulties in the numerical estimation of thin-walled structural elements are introduced by concentrated nature of the force impacts.
In the paper for bringing the three-dimensional problem for transversely isotropic plates to the two-dimensional one the expansion method of the required functions in Legendre polynomials from the normal coordinate is used. This approach allows us to take into account the transverse tangent lines and normal stresses. On the basis of this approach with the help of equations for transversely isotropic plates the method is developed to calculate the stress-strain state (SSS) under the action of concentrated force impacts.

2. LITERATURE REVIEW
Development of methods for constructing fundamental solutions (solutions corresponding to concentrated impacts), equations of the theory of thin elastic plates and shells are dedicated large number of publications. Problem statement, methods for their solutions and a number of concrete results are presented in monographs and scientific articles of S.A. Ambartsumyan [1], A.L. Gol'denveizer [2], W. Flügge [3], S. Lukasiewicz [4], as well as in a number of reviews of V.M. Darevsky [5], Y.P. Zhigalko [6] and others.
From the analysis of these studies we can conclude that there are two approaches to the construction of fundamental solutions of equations of thin elastic plates and shells.
The first is the study of singular solutions of homogeneous differential equations corresponding to a specific concentrated action.
Such an approach has been successfully used for the spherical shell by A.L. Gol'denveizer [7], for shallow spherical and cylindrical shells by I.N. Vekua [8], E. Reissner [9] and then for shallow shells of double curvature - N.A. Kiel [10], R. Ganowicz [11], A. Jahanshahi [12].
A major shortcoming of this approach is that for the definition of a singular solution, corresponding to a particular concentrated action, it is necessary to satisfy the system of geometric and static conditions in the vicinity of the singular point. Sometimes, especially for non-spherical shells, it leads to erroneous results, or to solutions containing excess regular solutions.
The second approach leads to the solution of differential equations with right-hand sides of the Dirac delta function. This used a variety of methods for constructing fundamental solutions. The most common of them - the Fourier integral method - developed in the works of P.M. Velichko, Y.A. Shevlyakova, V.K. Khizhnyak, V.P. Shevchenko [13 - 15], S. Lukasiewicz [16], J . Sanders [17], J. Simmonds [18].
All of these methods have been developed to study the nature of the component features of the stress-strain state of plates and shells, or to study the strength of plates and shells under local loads.
From this brief overview it is clear that for development of methods for solving boundary value problems in the theory of plates and shells the most suitable is a method of the two-dimensional Fourier integral. The theory of applying Fourier transform for solving partial differential equations with constant coefficients set forth in the works of V.S. Vladimirov [19] L. Schwartz [20] R. Edwards [21]. This method allows us to bring the system of resolving differential static equations of shallow plates and shells to a system of algebraic equations. Then the inverse Fourier transform restores the fundamental solution. Methods of treatment of the fundamental solution essentially depends on the type of the system of differential equations, so in this paper importance is given to the development of handling methods of integral transforms.

Thus, these examples illustrate the high effectiveness of the methods of fundamental solutions for the determination of the local SSS of thin elastic plates and shells with stress concentrators.

Expansion method in the thickness coordinate using Legendre polynomials is proposed in 1955 by I.N. Vekua [22]. Here, by the projection method are derived differential equations of equilibrium (movement) of prismatic shells of variable thickness with respect to moments of stress component. On the basis of Hooke's law for an isotropic body by multiplying the corresponding equations in Legendre polynomials, and integrating over the thickness coordinate relations are drawn between moments of stress components. Formulation of initial and boundary conditions is given. Method proposed in [22] spreads into thin shallow shells of variable thickness [13]. Cases of zero (N = 0) and the first (N = 1) approximation are considered in detail. Problems of existence and uniqueness of solutions of boundary value problems are studied. To a first approximation, a method is presented for constructing general solutions of the equations of equilibrium of plates, cylindrical and spherical shells [23].

A little later (in 1959) P. Chick published work [24], in which the method of relationships of the component stresses and displacements in series of Legendre polynomials is used to create the equilibrium equations of shells of revolution. The system of differential equations connecting the expansion coefficients of stress components is obtained from the variational principle of possible displacements. The second group of equations that establishes a connection between the coefficients of expansion of the components of stress and strain is derived by integrating by thickness of Hooke's law relationships.

For the construction of the equilibrium equations of isotropic plates by V.V Poniatowski [25] a method is used (proposed by E. Reissner earlier for the case of linear stress distribution through the thickness of the plate), according to which the tangential stress components are presented in a series of Legendre polynomials of the thickness coordinate, and the transverse stress components are determined from the equilibrium equations by integrating them in normal coordinate with the boundary conditions on the facial planes. Then, the variational principle of Castigliano is used. From the variational equations follow the equations of compatibility and appropriate boundary conditions. Proposed in [25] method is used to derive the equilibrium equations of anisotropic [26] and transversely isotropic plates. It also specifies the method of integrating the equations obtained.

Classical Kirchhoff-Love theory satisfactorily describes the SSS of relatively thin transversely isotropic plates, but does not account for phenomena caused by shifts and compression. On the other hand, the decision of problems of the theory of elasticity in the three-dimensional formulation leads to considerable mathematical difficulties. Therefore, the question of construction of refined theories is closely linked with the problem of bringing the three-dimensional problems to the two-dimensional ones.

Thus, the study on the basis of refined theories of SSS of transversely isotropic plates under the action of concentrated force impacts is actual scientific and technical problem.

3. THE PURPOSE AND OBJECTIVES OF THE STUDY

The purpose of this study is to develop the revised theory of plates, using the I.N. Vekua expansion method of unknown functions in Legendre polynomials from the transverse coordinate, in relation to the problem of determining SSS of transversely isotropic plates under the action of concentrated force impacts.

Achievement of an object provides:

– bringing the three-dimensional equations of elasticity theory to the two-dimensional ones by expanding of the unknown functions in Fourier series in Legendre polynomials with respect to the thickness coordinate;

– construction of fundamental solutions of these equations;

– study of the impact of the elastic parameters on the SSS of the plate.

In this paper we used the method of approximation of displacements, stresses and strains by Fourier series in Legendre polynomials on the transverse coordinate to output two-dimensional equations of elasticity theory for the transversely isotropic plates [27]. The fundamental solution of these equations is found using a two-dimensional Fourier integral and methods of treatment, built with a special G-function.

4. PROBLEM STATEMENT

We consider the transversely isotropic plate of thickness of $2h$ in a rectangular Cartesian coordinate system x, y, z.

Suppose that the concentrated force $\vec{F}$, applied at the coordinate origin (singular point) acts on the plate. Concentrated force can be represented as an abstraction (end-largest force) acting on a small area of the surface [28].

In solving problems about the action of concentrated forces, required SSS is considered as local that does not extend to the outer contour line of the plate. Therefore, we consider the plate as an infinite and assume that the required components of the SSS tend to zero at infinity. The validity of this assumption is checked after solving the problem.

The mathematical formulation of the problem consists in a complete system of equations of the elasticity theory without considering the boundary conditions on the edges of a real plate. The system of equilibrium equations of transversely isotropic plates is based on the theory of S.P. Timoshenko describing the SSS at bending and consists of [29]:

• geometrical relationships

$$e_{x1} = h\frac{\partial \gamma_x}{\partial x}, \qquad e_{xy1} = h\left(\frac{\partial \gamma_x}{\partial y} + \frac{\partial \gamma_y}{\partial x}\right),$$

$$e_{xz0} - \frac{e_{xz2}}{5} = \gamma_x + \frac{\partial w_0}{\partial x} \qquad (x \to y). \tag{1}$$

• Hooke's law relationships

$$M_x = D\left(e_{x1} + \nu e_{y1}\right),\ M_y = D\left(e_{y1} + \nu e_{x1}\right),\ H = \frac{1-\nu}{2} D e_{xy1},$$

$$Q_x = \Lambda\left(e_{xz0} - \frac{e_{xz2}}{5}\right) \qquad (x \to y), \tag{2}$$

where $D = \frac{2h^2}{3}\frac{E}{1-\nu^2}$, $\Lambda = \frac{5hG}{3}$.

• equilibrium equations

$$\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y} - Q_x + m_x = 0, \quad \frac{\partial M_y}{\partial y} + \frac{\partial H}{\partial x} - Q_y + m_y = 0,$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q_z = 0. \tag{3}$$

To find a fundamental solution of the system (1) – (3), the vector components of volumetric forces in formulas (3) should be taken in the form

$$m_x(x,y)=h^2 m_x^*\delta(x,y), \qquad m_y(x,y)=h^2 m_y^*\delta(x,y),$$
$$q_z(x,y)=h^2 q_z^*\delta(x,y) \qquad (x\to y), \tag{4}$$

where $m_x^*, m_y^*, q_z^* = const$, $\delta(x,y)$ – two-dimensional Dirac delta function [19].

5. DEFINITION OF TRANSFORMS OF GENERALIZED DISPLACEMENTS

Substituting geometrical ratio (1) in the equations of elasticity (2) and going to the dimensionless coordinate system $x_1 = x/h$, $x_2 = y/h$, $x_3 = z/h$, we obtain

$$M_1 = D_0\left(\frac{\partial\gamma_1}{\partial x_1}+\nu\frac{\partial\gamma_2}{\partial x_2}\right),\ M_2 = D_0\left(\frac{\partial\gamma_2}{\partial x_2}+\nu\frac{\partial\gamma_1}{\partial x_1}\right),$$
$$H=\frac{1-\nu}{2}D_0\left(\frac{\partial\gamma_1}{\partial x_2}+\frac{\partial\gamma_2}{\partial x_1}\right), \tag{5}$$
$$Q_1=\Lambda_0\left(\gamma_1+\frac{\partial w_0}{\partial x_1}\right), \qquad Q_2=\Lambda_0\left(\gamma_2+\frac{\partial w_0}{\partial x_2}\right),$$

where $D_0=\frac{D}{Eh^2}=\frac{2}{3}\frac{1}{1-\nu^2}$, $\Lambda_0=\frac{5G'}{3E}$.

Stretch bending and torque are determined in relation to the value Eh^2, and shear forces – in relation to the value Eh.

In the dimensionless coordinates we obtain:

$$\frac{\partial M_1}{\partial x_1}+\frac{\partial H}{\partial x_2}-Q_1+m_1=0,\ \frac{\partial M_2}{\partial x_2}+\frac{\partial H}{\partial x_1}-Q_2+m_2=0,$$
$$\frac{\partial Q_1}{\partial x_1}+\frac{\partial Q_2}{\partial x_2}+q_3=0, \tag{6}$$

where $m_1=m_1^*\delta(x_1,x_2)$, $m_2=m_2^*\delta(x_1,x_2)$, $q_3=q_3^*\delta(x_1,x_2)$.

Solving specified system, we obtain the transform of generalized displacements:

$$\tilde\gamma_1=\frac{1}{2\pi}\left[\frac{m_1^*}{D_0}\frac{\xi_1^2}{p^4}+3(1+\nu)m_1^*\frac{\xi_2^2}{p^2(p^2+a^2)}+\frac{q_3^*}{D_0}\frac{i\xi_1}{p^4}+\frac{m_2^*}{D_0}\frac{\xi_1\xi_2}{p^4}-\right.$$
$$\left.-3(1+\nu)m_2^*\frac{\xi_1\xi_2}{p^2(p^2+a^2)}\right], \tag{7}$$
$$\tilde\gamma_2=\frac{1}{2\pi}\left[\frac{m_2^*}{D_0}\frac{\xi_2^2}{p^4}+3(1+\nu)m_2^*\frac{\xi_1^2}{p^2(p^2+a^2)}+\frac{q_3^*}{D_0}\frac{i\xi_2}{p^4}+\frac{m_1^*}{D_0}\frac{\xi_1\xi_2}{p^4}-\right.$$
$$\left.-3(1+\nu)m_1^*\frac{\xi_1\xi_2}{p^2(p^2+a^2)}\right],$$
$$\tilde w_0=\frac{1}{2\pi}\left[-\frac{m_1^*}{D_0}\frac{i\xi_1}{p^4}-\frac{m_2^*}{D_0}\frac{i\xi_2}{p^4}+\frac{q_3^*}{D_0}\frac{1}{p^4}+\frac{q_3^*}{\Lambda_0}\frac{1}{p^2}\right],$$

where $p^2=\xi_1^2+\xi_2^2$, $a^2=3(1+\nu)\Lambda_0$; (ξ_1,ξ_2) – point coordinates in the space of transform.

6. DEFINITION OF GENERALIZED MOMENTS AND SHEAR FORCES IN THE SPACE OF TRANSFORMS

Applying the Fourier transform to equations of Hooke's law (2):

$$\tilde M_1=-D_0\left(i\xi_1\tilde\gamma_1+i\nu\xi_2\tilde\gamma_2\right),\ \tilde M_2=-D_0\left(i\xi_2\tilde\gamma_2+i\nu\xi_1\tilde\gamma_1\right),$$
$$\tilde H=-\frac{1-\nu}{2}D_0\left(i\xi_2\tilde\gamma_1+i\xi_1\tilde\gamma_2\right), \tag{8}$$
$$\tilde Q_1=\Lambda_0\left(\tilde\gamma_1-i\xi_1\tilde w_0\right),\ \tilde Q_2=\Lambda_0\left(\tilde\gamma_1-i\xi_2\tilde w_0\right).$$

Substitute the previously obtained transforms of generalized displacements (7) to the transforms of bending moments, torque and shear forces (8):

$$\tilde M_1=-\frac{1}{2\pi}\left[m_1^*\frac{\xi_1^3}{p^4}+2m_1^*\frac{i\xi_1\xi_2^2}{p^2(p^2+a^2)}-q_3^*\frac{\xi_1^2}{p^4}+m_2^*\frac{i\xi_1^2\xi_2}{p^4}-\right.$$
$$\left.-2m_2^*\frac{i\xi_1^2\xi_2}{p^2(p^2+a^2)}+m_2^*\nu\frac{\xi_2^3}{p^4}-q_3^*\nu\frac{\xi_2^2}{p^4}+m_1^*\nu\frac{i\xi_1\xi_2^2}{p^4}\right],$$
$$\tilde M_2=-\frac{1}{2\pi}\left[m_2^*\frac{\xi_2^3}{p^4}+2m_2^*\frac{i\xi_1^2\xi_2}{p^2(p^2+a^2)}-q_3^*\frac{\xi_2^2}{p^4}+m_1^*\frac{i\xi_1\xi_2^2}{p^4}-\right.$$
$$\left.-2m_1^*\frac{i\xi_1\xi_2^2}{p^2(p^2+a^2)}+m_1^*\nu\frac{\xi_1^3}{p^4}-q_3^*\nu\frac{\xi_1^2}{p^4}+m_2^*\nu\frac{i\xi_1^2\xi_2}{p^4}\right],$$
$$\tilde H=-\frac{1}{2\pi}\left[(1-\nu)m_1^*\frac{i\xi_1^2\xi_2}{p^4}+m_1^*\frac{i\xi_2^3}{p^2(p^2+a^2)}-\right. \tag{9}$$
$$-(1-\nu)q_3^*\frac{\xi_1\xi_2}{p^4}+(1-\nu)m_2^*\frac{i\xi_1\xi_2^2}{p^4}-m_2^*\frac{i\xi_1\xi_2^2}{p^2(p^2+a^2)}+$$
$$\left.+m_2^*\frac{i\xi_1^3}{p^2(p^2+a^2)}-m_1^*\frac{i\xi_1^2\xi_2}{p^2(p^2+a^2)}\right],$$
$$\tilde Q_1=\frac{a^2}{2\pi}\left[m_1^*\frac{\xi_2^2}{p^2(p^2+a^2)}-m_2^*\frac{\xi_1\xi_2}{p^2(p^2+a^2)}\right]-\frac{q_3^*}{2\pi}\frac{i\xi_1}{p^2},$$
$$\tilde Q_2=\frac{a^2}{2\pi}\left[m_2^*\frac{\xi_1^2}{p^2(p^2+a^2)}-m_1^*\frac{\xi_1\xi_2}{p^2(p^2+a^2)}\right]-\frac{q_3^*}{2\pi}\frac{i\xi_2}{p^2}.$$

Let us denote

$$\tilde\Phi_1(\xi_1,\xi_2)=\frac{i\xi_1^3}{p^4},\ \tilde\Phi_2(\xi_1,\xi_2)=\frac{i\xi_1^2\xi_2}{p^4},$$
$$\tilde\Phi_3(\xi_1,\xi_2)=\frac{i\xi_1^2\xi_2}{p^2(p^2+a^2)},\ \tilde\Phi_5(\xi_1,\xi_2)=\frac{\xi_1}{p^2}, \tag{10}$$
$$\tilde\Phi_4(\xi_1,\xi_2)=\frac{i\xi_1^3}{p^2(p^2+a^2)},\ \tilde\Phi_6(\xi_1,\xi_2)=\frac{\xi_1^2}{p^4},$$
$$\tilde\Phi_7(\xi_1,\xi_2)=\frac{\xi_1\xi_2}{p^4},\ \tilde\Phi_8(\xi_1,\xi_2)=\frac{i\xi_1^2}{p^2(p^2+a^2)},$$
$$\tilde\Phi_9(\xi_1,\xi_2)=\frac{\xi_1\xi_2}{p^2(p^2+a^2)}.$$

Then, bending moments in the space of transforms are written as

$$\tilde{M}_1 = -\frac{1}{2\pi}\Big[m_1^*\tilde{\Phi}_1(\xi_1,\xi_2) + 2m_1^*\tilde{\Phi}_3(\xi_2,\xi_1) - q_3^*\tilde{\Phi}_6(\xi_1,\xi_2) + m_2^*\tilde{\Phi}_2(\xi_1,\xi_2) - 2m_2^*\tilde{\Phi}_3(\xi_1,\xi_2) + m_2^*\nu\tilde{\Phi}_1(\xi_2,\xi_1) - q_3^*\nu\tilde{\Phi}_6(\xi_2,\xi_1) + m_1^*\nu\tilde{\Phi}_2(\xi_2,\xi_1)\Big],$$

$$\tilde{M}_2 = -\frac{1}{2\pi}\Big[m_2^*\tilde{\Phi}_1(\xi_2,\xi_1) + 2m_2^*\tilde{\Phi}_3(\xi_1,\xi_2) - q_3^*\tilde{\Phi}_6(\xi_2,\xi_1) + m_1^*\tilde{\Phi}_2(\xi_2,\xi_1) - 2m_1^*\tilde{\Phi}_3(\xi_2,\xi_1) + m_1^*\nu\tilde{\Phi}_1(\xi_1,\xi_2) - q_3^*\nu\tilde{\Phi}_6(\xi_1,\xi_2) + m_2^*\nu\tilde{\Phi}_2(\xi_1,\xi_2)\Big],$$

$$\tilde{H} = -\frac{1}{2\pi}\Big[(1-\nu)m_1^*\tilde{\Phi}_2(\xi_1,\xi_2) + m_1^*\tilde{\Phi}_4(\xi_2,\xi_1) - (1-\nu)q_3^*\tilde{\Phi}_7(\xi_1,\xi_2) + (1-\nu)m_2^*\tilde{\Phi}_2(\xi_2,\xi_1) - m_2^*\tilde{\Phi}_3(\xi_2,\xi_1) + m_2^*\tilde{\Phi}_4(\xi_1,\xi_2) - m_1^*\tilde{\Phi}_3(\xi_1,\xi_2)\Big], \quad (11)$$

$$\tilde{Q}_1 = \frac{a^2}{2\pi}\Big[m_1^*\tilde{\Phi}_8(\xi_2,\xi_1) - m_2^*\tilde{\Phi}_9(\xi_1,\xi_2)\Big] - \frac{q_3^*}{2\pi}\tilde{\Phi}_5(\xi_1,\xi_2),$$

$$\tilde{Q}_2 = \frac{a^2}{2\pi}\Big[m_2^*\tilde{\Phi}_8(\xi_1,\xi_2) - m_1^*\tilde{\Phi}_9(\xi_1,\xi_2)\Big] - \frac{q_3^*}{2\pi}\tilde{\Phi}_5(\xi_2,\xi_1).$$

You must now turn ratio (11), to find the original of internal power factors.

7. FINDING THE ORIGINAL OF INTERNAL POWER FACTORS

First, we find the originals of functions (10) using a Fourier integral [30]

$$F^{-1}\Big[\tilde{f}(\xi_1,\xi_2)\Big] = f(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}(\xi_1,\xi_2)e^{-i(\xi_1 x_1+\xi_2 x_2)}d\xi_1 d\xi_2. \quad (12)$$

Obtain

$$\Phi_1(x_1,x_2) = \frac{x_1(x_1^2+3x_2^2)}{2(x_1^2+x_2^2)^2},\ \Phi_2(x_1,x_2) = -\frac{x_2(x_1^2-x_2^2)}{2(x_1^2+x_2^2)^2},$$

$$\Phi_3(x_1,x_2) = \frac{x_2}{2(x_1^2+x_2^2)}G_{0,1}\left(a\sqrt{x_1^2+x_2^2}\right) + \frac{x_2(3x_1^2-x_2^2)}{2(x_1^2+x_2^2)^2}G_{1,2}\left(a\sqrt{x_1^2+x_2^2}\right), \quad (13)$$

$$\Phi_4(x_1,x_2) = \frac{3x_1}{2(x_1^2+x_2^2)}G_{0,1}\left(a\sqrt{x_1^2+x_2^2}\right) + \frac{x_1(x_1^2-3x_2^2)}{2(x_1^2+x_2^2)^2}G_{1,2}\left(a\sqrt{x_1^2+x_2^2}\right),$$

$$\Phi_5(x_1,x_2) = \frac{x_1}{x_1^2+x_2^2},\ \Phi_7(x_1,x_2) = -\frac{1}{2}\frac{x_1x_2}{x_1^2+x_2^2},$$

$$\Phi_6(x_1,x_2) = -\frac{1}{2}\ln\frac{\gamma\sqrt{x_1^2+x_2^2}}{2} - \frac{1}{4}\frac{x_1^2-x_2^2}{x_1^2+x_2^2},$$

$$\Phi_8(x_1,x_2) = \frac{1}{2}\left(G_{0,0}\left(a\sqrt{x_1^2+x_2^2}\right) + \frac{x_1^2-x_2^2}{x_1^2+x_2^2}\times G_{1,1}\left(a\sqrt{x_1^2+x_2^2}\right)\right),\ \Phi_9(x_1,x_2) = \frac{x_1x_2}{x_1^2+x_2^2}G_{1,1}\left(a\sqrt{x_1^2+x_2^2}\right).$$

where $G_{n,\nu}(rz)$ – a special G-function [31].

Applying inversion formula for the two-dimensional Fourier integral (12) to the transforms of internal power factors (11) and taking into account the expression (13), we write expressions for M_1, M_2, H, Q_1, Q_2 in the space of the originals

$$M_1 = -\frac{1}{2\pi}\left[m_1^*\frac{x_1(x_1^2+3x_2^2)}{2(x_1^2+x_2^2)^2} + 2m_1^*\left\{\frac{x_1}{2(x_1^2+x_2^2)}G_{0,1}\left(a\times\sqrt{x_1^2+x_2^2}\right) - \frac{x_1(x_1^2-3x_2^2)}{2(x_1^2+x_2^2)^2}G_{1,2}\left(a\sqrt{x_1^2+x_2^2}\right)\right\} + q_3^*\times\left\{\frac{1}{2}\ln\frac{\gamma\sqrt{x_1^2+x_2^2}}{2} + \frac{1}{4}\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\right\} - m_2^*\frac{x_2(x_1^2-x_2^2)}{2(x_1^2+x_2^2)^2} - 2m_2^*\left\{\frac{x_2}{2(x_1^2+x_2^2)}G_{0,1}\left(a\sqrt{x_1^2+x_2^2}\right) + \frac{x_2(3x_1^2-x_2^2)}{2(x_1^2+x_2^2)^2}\times G_{1,2}\left(a\sqrt{x_1^2+x_2^2}\right)\right\} + m_2^*\nu\frac{x_2(3x_1^2+x_2^2)}{2(x_1^2+x_2^2)^2} + q_3^*\nu\left\{\frac{1}{2}\ln\frac{\gamma\sqrt{x_1^2+x_2^2}}{2} + \frac{1}{4}\frac{x_2^2-x_1^2}{x_1^2+x_2^2}\right\} + m_1^*\nu\frac{x_1(x_1^2-x_2^2)}{2(x_1^2+x_2^2)^2}\right],$$

$$M_2 = -\frac{1}{2\pi}\left[m_2^*\frac{x_2(3x_1^2+x_2^2)}{2(x_1^2+x_2^2)^2} + 2m_2^*\left\{\frac{x_2}{2(x_1^2+x_2^2)}G_{0,1}\left(a\times\sqrt{x_1^2+x_2^2}\right) + \frac{x_2(3x_1^2-x_2^2)}{2(x_1^2+x_2^2)^2}G_{1,2}\left(a\sqrt{x_1^2+x_2^2}\right)\right\} + q_3^*\times\left\{\frac{1}{2}\ln\frac{\gamma\sqrt{x_1^2+x_2^2}}{2} + \frac{1}{4}\frac{x_2^2-x_1^2}{x_1^2+x_2^2}\right\} + m_1^*\frac{x_1(x_1^2-x_2^2)}{2(x_1^2+x_2^2)^2} - 2m_1^*\left\{\frac{x_1}{2(x_1^2+x_2^2)}G_{0,1}\left(a\sqrt{x_1^2+x_2^2}\right) - \frac{x_1(x_1^2-3x_2^2)}{2(x_1^2+x_2^2)^2}\times G_{1,2}\left(a\sqrt{x_1^2+x_2^2}\right)\right\} + m_1^*\nu\frac{x_1(x_1^2+3x_2^2)}{2(x_1^2+x_2^2)^2} + q_3^*\nu\left\{\frac{1}{2}\ln\frac{\gamma\sqrt{x_1^2+x_2^2}}{2} + \frac{1}{4}\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\right\} - m_2^*\nu\frac{x_2(x_1^2-x_2^2)}{2(x_1^2+x_2^2)^2}\right],$$

$$H = -\frac{1}{2\pi(x_1^2+x_2^2)}\left[-(1-\nu)m_1^*\frac{x_2(x_1^2-x_2^2)}{2(x_1^2+x_2^2)} + m_1^*\times\right.$$

$$\times\left\{x_2 G_{0,1}\left(a\sqrt{x_1^2+x_2^2}\right)-\frac{x_2\left(3x_1^2-x_2^2\right)}{x_1^2+x_2^2}G_{1,2}\left(a\sqrt{x_1^2+x_2^2}\right)\right\}+$$

$$+(1-\nu)q_3^*\frac{x_1x_2}{2}+(1-\nu)m_2^*\frac{x_1\left(x_1^2-x_2^2\right)}{2\left(x_1^2+x_2^2\right)}+ \quad (14)$$

$$+m_2^*\left\{x_1G_{0,1}\left(a\sqrt{x_1^2+x_2^2}\right)+\frac{x_1\left(x_1^2-3x_2^2\right)}{x_1^2+x_2^2}G_{1,2}\left(a\sqrt{x_1^2+x_2^2}\right)\right\}\Bigg]\Bigg],$$

$$Q_1=\frac{a^2}{2\pi}\left[\frac{m_1^*}{2}\left\{G_{0,0}\left(a\sqrt{x_1^2+x_2^2}\right)+\frac{x_2^2-x_1^2}{x_1^2+x_2^2}G_{1,1}\left(a\sqrt{x_1^2+x_2^2}\right)\right\}-\right.$$

$$\left.-m_2^*\frac{x_1x_2}{x_1^2+x_2^2}G_{1,1}\left(a\sqrt{x_1^2+x_2^2}\right)\right]-\frac{q_3^*}{2\pi}\frac{x_1}{x_1^2+x_2^2},$$

$$Q_2=\frac{a^2}{2\pi}\left[\frac{m_2^*}{2}\left\{G_{0,0}\left(a\sqrt{x_1^2+x_2^2}\right)+\frac{x_1^2-x_2^2}{x_1^2+x_2^2}G_{1,1}\left(a\sqrt{x_1^2+x_2^2}\right)\right\}-\right.$$

$$\left.-m_1^*\frac{x_1x_2}{x_1^2+x_2^2}G_{1,1}\left(a\sqrt{x_1^2+x_2^2}\right)\right]-\frac{q_3^*}{2\pi}\frac{x_2}{x_1^2+x_2^2}.$$

We see that the expressions obtained for the moments have a singularity in the form of a logarithmic singularity. The expressions for the shear forces have the same feature, which G -function. Obtained solutions can then be used as the core of integral representations in problems of equilibrium of plates weakened by defects such as cracks and fine inclusions.

8. THE RESULTS OF STUDIES OF THE ELASTIC PARAMETERS INFLUENCE ON THE PLATE SSS USING THE DEVELOPED TECHNIQUE

To study the features of the SSS of transversely isotropic plates under concentrated force action set: $m_1^*=m_2^*=q_3^*=1$.

Results are presented in a dimensionless Cartesian coordinate system x_1, x_2.

Numerical studies were carried out for the following materials of plates: lead and zinc. Poisson's ratios (ν) for these materials are 0.446 and 0.212, respectively [32]. Parameter of sliding compliance is $\frac{E}{G'}=2{,}6$.

Fig. 1–3 shows graphs of changes of generalized moments M_1, M_2, H $\left(\frac{M_1}{10^9},\frac{M_2}{10^9},\frac{H}{10^9}\right)$ along the abscissa axis These graphs show that with decreasing of Poisson's ratio values of generalized moments increase.

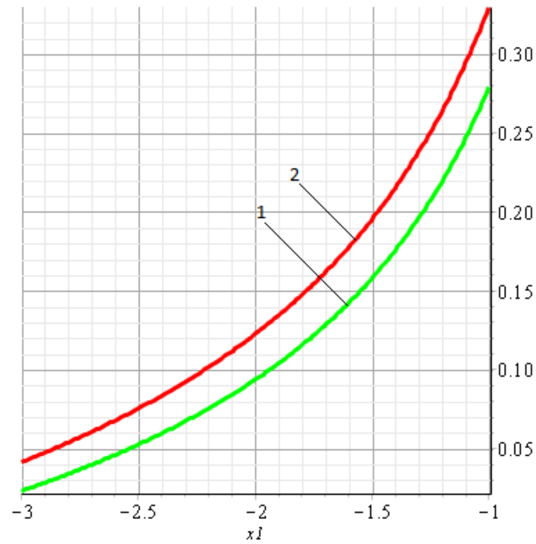

Figure 1. The bending moment M_1: 1 - material is lead; 2 - material is zinc

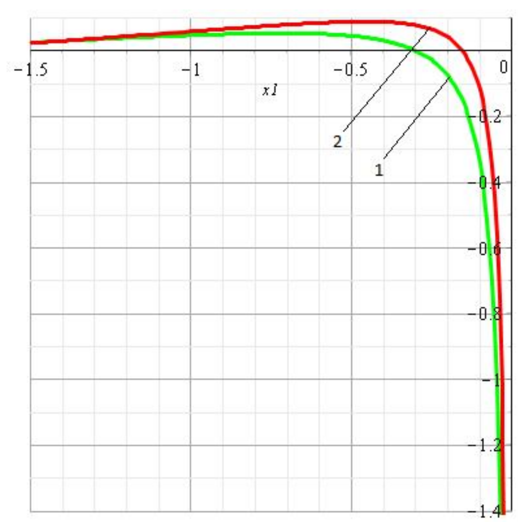

Figure 2. The bending moment M_2: 1 - material is lead; 2 - material is zinc

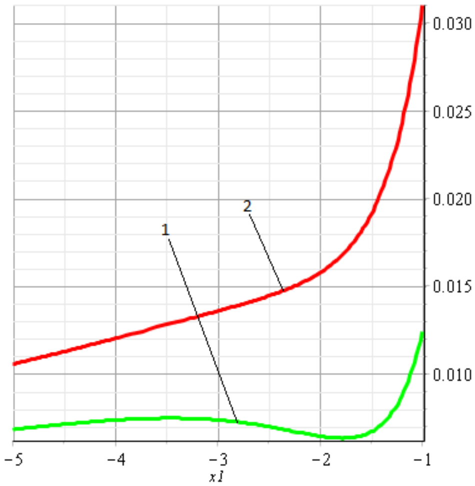

Figure 3. Torque H: 1 - material is lead; 2 - material is zinc

Fig. 4 and 5 shows graphs of the generalized forces $Q_1, Q_2\left(\frac{Q_1}{10^9}, \frac{Q_2}{10^9}\right)$ respectively. These graphs show the independence of the generalized forces Q_1, Q_2 from the elastic constants.

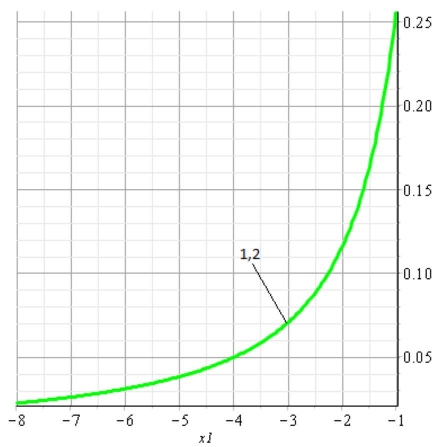

Figure 4. Shear force Q_1: 1 - material is lead; 2 - material is zinc

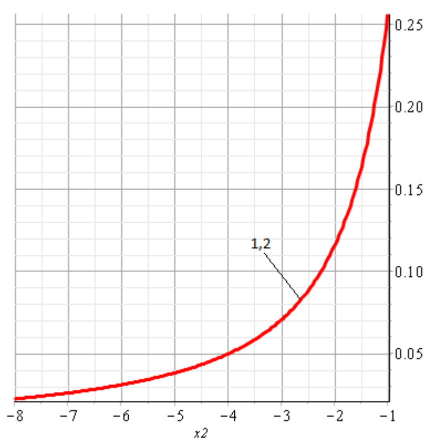

Figure 5. Shear force Q_2: 1 - material is lead; 2 - material is zinc

These studies allow investigating analytically the behavior of internal force values depending on the elastic constants of different materials and investigating the nature of the special moments and shearing forces in the concentrated actions.

9. DISCUSSION OF RESULTS OBTAINED BY THIS METHOD

As noted above, the classical Kirchhoff-Love theory satisfactorily describes the SSS of relatively thin transversely isotropic plates, but does not account for phenomena caused by shifts and compression.

The developed method allows the calculation of generalized shear forces and bending moments for plates subjected to the action of a force applied at the origin of the coordinate system. In contrast to the studies carried out by other authors, here we use the equilibrium equations of transversely isotropic plates on the basis of the theory of S.P. Timoshenko describing the SSS at a bend. This makes it possible to consider the shells and plates which have a thickness of about 1/5 with respect to the characteristic size.

Because in reality, the forces acting on the structural elements are always distributed (perhaps on a very small, but finite domains), the results obtained in this study are preliminary.

The resulting fundamental solution will enable to solve a number of new problems of bending of plates of medium thickness. In the presence of concentrated dislocations the fundamental solutions - Green's functions - are the foundation for building potential representations as integrals of displacement jumps distributed with unknown density. Such integral representations can be used in solving the problems of bending of plates with different kinds of defects, cuts and incisions. Due to the large depreciation of the energy, refinery and chemical equipment in Ukraine at the moment, particularly relevant issue is the problem of extending the working life of equipment, even in the presence of micro-defects in it. Calculation of the strength of such elements by using the theory of shells and plates of medium thickness in the presence of different kinds of defects based on the obtained fundamental solutions will facilitate the adjustment of the terms of reserve maintenance periods, priority of replacement of worn-out equipment.

The practical significance of these results is the possibility of using the developed methods for solving problems in the calculations associated with the design and definition of the operating parameters of thin-walled structural elements of transversely isotropic materials under the action of concentrated force impacts. The results can be used in scientific research institutes, design organizations and other research institutions involved in the calculations of thin-walled structural elements.

10. CONCLUSIONS

Fundamental solution of equations of statics transversely isotropic plates on the basis of a generalized theory was constructed.

Achievement of this objective includes bringing the three-dimensional equations of elasticity theory to the two-dimensional ones by expanding the unknown functions in Fourier series in Legendre polynomials with respect to the thickness coordinate. This approach allowed us to take into account the transverse shear and normal stresses. Since the classical theory of Kirchhoff-Love does not take into account these stresses, the study on the basis of refined theories of the SSS of transversely isotropic plates under the action of concentrated force impacts is an actual scientific and technical problem. The fundamental solution of these equations was found using a two-dimensional Fourier integral and methods of treatment, built with a special G-function.

Numerical studies of the SSS of transversely isotropic plates are carried out. These studies allowed revealing patterns of behavior of the SSS components, depending on the elastic constants of transversely isotropic material.
The resulting fundamental solution makes it possible to solve a number of new tasks of bending of plates of medium thickness. In the presence of concentrated dislocations fundamental decisions - Green's functions - are the foundation for building potential representations as integrals of displacement jumps distributed with unknown density. Such integral representations can be used in solving the problems of bending of plates with different kinds of defects, cuts and incisions.

REFERENCES

[1] Ambartsumyan, S. A. 1974. The general theory of anisotropic shells. M.: Science, 446.

[2] Goldenveizer, A. L. 1944. Research of the stress condition of the spherical shell. Applied Mathematics and Mechanics, Vol. 6, 441–467.

[3] Flügge, W. 1966. Concentrated forces on shells. Proceedings of Xlth Internat. Congress of Applied Mechanics. Munich: Springer Verlag, 270–276.

[4] Łukasiewicz, S. A. 1976. Introduction of concentrated loads in plates and shells. Progress in Aerospace Sciences, Vol. 17, 109–146.

[5] Darevskii, V. M. 1966. Contact problems of the theory of shells (the action of local loads on the shell). Proceedings of VI All-Union Conference on the theory of shells and plates. M.: Nauka, 927–933.

[6] Zhigalko, Y. P. 1966. Calculation of thin elastic cylindrical shells at local loads (literature review, methods and results). Investigations in the theory of plates and shells, Vol. 4, 3–41.

[7] Goldenveizer, A. L. 1954. On the question of the calculation of shells on concentrated forces. Applied Mathematics and Mechanics, Vol. 8, № 2, 181–186.

[8] Vekua, I. N. 1970. Variational principles of the theory of shells. Publisher of Tbilisi, 300.

[9] Reissner, E. 1985. Reflections on the theory of elastic plates. Applied Mechanics Reviews, Vol. 38, № 11, 1453–1464.

[10] Kiel, N. A. 1973. On the action of local loads on the shell. Proceedings of the universities. Construction and architecture, № 3, 43–46.

[11] Ganowicz, R. 1973. On fundamental singularity in the theory of shallow cylindrical shells. Arch. mech. Stosowanej, Vol. 25, № 6, 985–992.

[12] Jahanshahi, A. 1963. Force Singularities of Shallow Cylindrical Shells. Journal of Applied Mechanics. ASME International, Vol. 30, № 3, 342–345.

[13] Velichko, P. M., Khizhnyak, V. K., Shevchenko, V. P. 1971. Deformation of shells of positive curvature in the concentrated actions. Stress concentration, Vol. 3, 31.

[14] Velichko, P. M., Khizhnyak, V. K., Shevchenko, V. P. 1968. Research local stresses in plates and shells under concentrated load. Proceedings of III All-Union Congress on theor. mechanics. M.: Publishing House of the USSR Academy of Sciences, 67.

[15] Velichko, P. M., Khizhnyak, V. K., Shevchenko, V. P. 1975. Local stresses in the shells of positive, neutral and negative curvature. Proceedings of X All-Union Conf. on the theory of shells and plates. Vol. 1. Tbilisi: Publishing house Mitsniereba, 31–41.

[16] Lukasiewicz, S. 1979. Local loads in plates and shells. Alpen aan den Rijn, Sijthoff and Noordhoff. Warszawa: PWM, 569.

[17] Sanders, J. L. 1970. Singular Solutions to the Shallow Shell Equations. Journal of Applied Mechanics, Vol. 37, № 2, 361–366.

[18] Simmonds, J. G., Bradley, M. R. 1976. The Fundamental Solution for a Shallow Shell With an Arbitrary Quadratic Midsurface. Journal of Applied Mechanics, Vol. 43, № 2, 286–290.

[19] Vladimirov, V. S. 1976. Generalized functions in mathematical physics. M.: Science, 280.

[20] Schwartz, L. 1965. Mathematical Methods for the Physical Sciences. M.: World, 412.

[21] Edwards, R. 1969. Function analysis. M.: World, 1071.

[22] Vekua, I. N. 1955. A method for calculating prismatic shells. Pr. Tbilis. Mat. Inst., № 21, 191–253.

[23] Vekua, I. N. 1965. The theory of thin shallow shells of variable thickness. Tbilisi: Metsniereba, 103.

[24] Cicala, P. 1959. Sulla teoria elastica della plate sottile. Gorn. Genio civile, Vol. 97, № 4, 238–256.

[25] Poniatowski, V. V. 1962. On the theory of plates of medium thickness. Applied Mathematics and Mechanics, Vol. 26, № 2, 335–341.

[26] Poniatowski, V. V. 1964. On the theory of bending of anisotropic plates. Applied Mathematics and Mechanics, Vol. 28, № 6, 1033–1039.

[27] Bokov, I. P., Strelnikova, E. A. 2015. Construction of fundamental solutions static equation isotropic plates of medium thickness. Eastern-European Journal of Enterprise Technologies, Vol. 4, № 76, 27–34. DOI: 10.15587/1729-4061.2015.47232

[28] Han, H. 1988. Elasticity Theory: Fundamentals of linear theory and its application. M.: World, 344.

[29] Pelekh, B. L., Lazko, V. A. 1982. Laminated anisotropic plates and shells with stress concentrators. K.: Science thought, 296.

[30] Sneddon, I. 1955. Fourier transform. M.: Foreign literature publishing house, 668.

[31] Khizhnyak, V. K., Shevchenko, V. P. 1980. Mixed problem in the theory of plates and shells. DonGU: Donetsk, 128.

[32] Dementyev, A. D., Nazarov, L. A., Nazarova, L. A. 2002. Applications of the theory of elasticity. Novosibirsk, 224.