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Construction of Doubly Linked Block Association Scheme and Its Block Design

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ABSTRACT- In the sphere of Partially Balanced Incomplete Block (PBIB) Design, the Doubly Linked Block (DLB) Association Scheme is a novel association scheme. Its known solutions are very limited in numbers and countable on the finger's tip. This paper presents a new DLB association scheme and a new PBIB design based on the DLB association schemes.

KEYWORDS- Association Scheme; Partially Balanced Incomplete Block Design; Dual Design.

I. INTRODUCTION

In the design of experiments, Balanced Incomplete Block Design (BIBD) has several advantages, but BIBDs do not exist for any parameter combinations. This led Bose and Nair in 1939 [1] to introduce a class of binary, equireplicate, and proper designs, which are called PBIB designs. The definition of PBIB designs was generalized by Nair and Rao [2] in 1942, which includes the cubic and other higher dimensional lattices. Later on, it was realized that the existence of PBIB designs depends on an abstract relationship configuration, so-called Association Scheme. Further, introducing the concept of association schemes, PBIB designs based on the association schemes, *viz.,* Group Divisible, Triangular, Latin Square, Cyclic and Singly Linked Block (SLB) association schemes, with the specific range of parameters $r\leq 10$ and $k\leq 10$ due to Bose and Shimamoto [3] are available. Shrikhande (1950) [4] introduced the concept of the Singly Linked Block association scheme, defining the association relationship of two blocks based on their intersection number of treatment which must be either 0 or 1.

Youden (1951) [5] planned that the dual of a BIBD having the parameters $v^*, b^*, r^*, k^*, \lambda^*$ are so called "$\lambda^* -$ linked blocks" because every pair of blocks of the dual design has λ^* treatments in common and the dual of a BIBD are not necessarily always partially balanced. From the works of Shrikhande (1952) [6] it is learnt that the dual of a BIBD $(v^*, b^*, r^*, k^*, \lambda^*)$ will be partially balanced if $\lambda^* = 1$ or $r^* = k, k^* = k-2, \lambda^* = 2$ where k is the block-size of the dual design.

Association Scheme:

Given v treatments, viz., 1, 2, …, v, a relation satisfying the following conditions is said to be an association scheme with m classes:

- Any two treatments are either 1^{st}, 2^{nd}, …, or m^{th} associates, the relation of the association being symmetrical, i.e., if the treatment α is an i^{th} associate of the treatment β, then β is also an i^{th} associate of the treatment α (i=1, 2, …, m).
- Given a treatment α, the number of i^{th} associates of α is n_i, the number n_i being independent of α (i=1, 2, ..., m).
- If any two treatments α and β are i^{th} associates to each other, then the number of treatments which are the j^{th} associates of α and k^{th} associates of β is p^i_{jk} and is independent of the pair of i^{th} associates α and β (i,j,k=1, 2, …., m).

The numbers v, n_i, p^i_{jk} (i, j, k=1, 2, …, m), are the parameters of the association scheme. They hold good for the following relations, given by Bose and Nair: $\sum_{i=1}^{m} n_i$=v-1, $\sum_{k=1}^{m} p^i_{jk} = \begin{cases} n_i - 1, & if\ i = j \\ n_j, & otherwise \end{cases}$ and $n_i p^i_{jk} = n_j p^j_{ik} = n_k p^k_{ij}$.

Partially Balanced Incomplete Block Design:

Given an m-association scheme with the parameters v, n_i, p^i_{jk} (i, j, k=1, 2, …, m), a PBIB design based on the given association scheme is an arrangement of v treatments in b blocks, each of size k ($< v$), such that:

(i) Each treatment appears at most once in a block,
(ii) Each treatment gets replicated r times,
(iii) Any two i^{th} associate treatments occur in λ_i blocks.

Dual Design:

The dual design D' of a block design D (v, b, r, k) is a design given by interchanging the role of blocks and treatments. Then, the parameters of D' are $v' = b$, $b' = v$, $r' = k$, $k' = r$.

II. DOUBLY LINKED BLOCK (DLB) ASSOCIATION SCHEME

DLB Association Scheme starting from a BIBD with pairwise balancing parameter value 2 i.e., $\lambda = 2$ is due to Meitei (2015) [7]. Here, the proposed DLB Association Scheme is modified in more general, in the sense that the starting BIBD for construction of the DLB Association Scheme is replaced with Incomplete Block Design (IBD) as follows:

For a given IBD, D with parameters v, b, r_i, (i=1,2,3…, v) such that-

- any two blocks intersect either one or two treatment(s) in common,
- any block has n_i blocks which intersect i (=1, 2) treatment(s) in common and no other blocks,
- any two blocks having one(two) treatment(s) in common, have exactly *t(s)* blocks each of which intersects with each of the two blocks exactly one treatment in common, define two block numbers of *D* to be first associates if they have exactly one treatment in common and second associates if they have exactly two treatments in common. Then this association rule is an association scheme with two classes, which may be called the DLB Association Scheme.

A. Construction of DLB Association Scheme:

A result of the construction of the DLB Association Scheme, starting from a BIBD is given as follows:

Theorem 2.1:

If there exists a BIBD, D $(v^*, b^*, r^*, k^*, \lambda^*)$ satisfying the three conditions of DLB Association Scheme (given in section 2) implies the existence of DLB Association Scheme with parameters: $v = b^*$, $n_1 = b^* - n_2 - 1 = b^* - (\lambda^* - 1)\binom{k^*}{2} - 1$, $n_2 = (\lambda^* - 1)\binom{k^*}{2}$, $P^1_{11} = t$, $P^2_{11} = s$.

Proof: By the definition of DLB Association Scheme, the block numbers of the given BIBD will be the treatment numbers of the DLB Association Scheme. Therefore, $v = b^*$ and since any pairs of treatment occurs in λ^* blocks, so, each of the $\binom{k^*}{2}$ pairs of treatment from a block of the given design *D*, occur $(\lambda^* - 1)$ in some $(\lambda^* - 1)\binom{k^*}{2}$ other blocks separately by conditions *(i)* and *(ii)* of DLB Association Scheme. So, any block B_j (say) has $(\lambda^* - 1)\binom{k^*}{2}$ other blocks intersecting two treatments in common with B_j i.e., any treatment of the Association Scheme corresponding to a block number of the parent design has $(\lambda^* - 1)\binom{k^*}{2}$ second associates. Further, since any two blocks of *D* intersect either one or two treatment(s) in common, then each of the remaining $b^* - (\lambda^* - 1)\binom{k^*}{2} - 1$ blocks intersect 1 treatment in common with B_j i.e., any treatment of the association scheme corresponding to a block number of the parent design have $b^* - (\lambda^* - 1)\binom{k^*}{2} - 1$ first associates. Thus, $n_1 = b^* - (\lambda^* - 1)\binom{k^*}{2} - 1$ and $n_2 = (\lambda^* - 1)\binom{k^*}{2}$. From the condition (iii) of the definition of DLB Association Scheme i.e., any two blocks B_j^* and $B_{j'}^*$, *(j≠j')* (say) having one(two) treatment(s) in common, have exactly *t(s)* blocks each of which intersects with each of the two blocks B_j^* and $B_{j'}^*$*(j≠j')* exactly one treatment in common. So, it is evident that any two treatments of the association scheme corresponding to the two blocks B_j^* and $B_{j'}^*$, *(j≠j')* of the parent design which are first associates to one another (i.e., $|B_j^* \cap B_{j'}^*|=1$) have 't' treatments in common which are first associates to both the treatments i.e., P^1_{11}= t. And any two second associates of the association scheme corresponding to the two blocks B_j^{**} and $B_{j'}^{**}$ of the parent design such that $|B_j^{**} \cap B_{j'}^{**}|=2$ have 's' treatments in common which are first associates to both the treatments i.e.,P^2_{11}= s. Hence, the parameters of the DLB Association Scheme are $v = b^*$, $n_1 = b^* - n_2 - 1 = b^* - (\lambda^* - 1)\binom{k^*}{2} - 1$, $n_2 = (\lambda^* - 1)\binom{k^*}{2}$, $P^1_{11} = t$, $P^2_{11} = s$.

Using the BIBD, serial number 4.4.3, Dey, A. (1986) [8], an example illustrating the Theorem 2.1 is given as follows:

Example 2.1**:**

The BIBD has the parameters *v*, $b=\binom{v}{k}$, $r=\binom{v-1}{k-1}$, *k*, $\lambda=\binom{v-2}{k-2}$. Taking $v = 5$ and $k = 3$, the parameters of the BIBD become $v^* = 5$, $b^* = 10$, $r^* = 6$, $k^* = 3$, $\lambda^* = 3$. The following is a plan of BIBD with the parameters v^*=5, b^*=10, r^*=6, k^*=3, λ^*= 3, where 1, 2, 3, 4, 5 represent the treatments and parentheses indicate the blocks. B_1 = (1, 2, 3), B_2 = (1, 2, 4), B_3 = (1, 2, 5), B_4 = (1, 3, 4), B_5 = (1, 3, 5), B_6 = (1, 4, 5), B_7 = (2, 3, 4), B_8 = (2, 3, 5), B_9 = (2, 4, 5), B_{10} = (3, 4, 5). Then, by the Theorem 2.1, the number of treatments of the DLB association scheme is *v* = 10.

The table below describes the first and second associates of all 10 treatments.

Table 1: Association Table

Treatments	First associates	Second associates
1	6,9,10	2,3,4,5,7,8
2	5,8,10	1,3,4,6,7,9
3	4,7,10	1,2,5,6,8,9
4	3,8,9	1,2,5,6,7,10
5	2,7,9	1,3,4,6,8,10
6	1,7,8	2,3,4,5,9,10
7	3,5,6	1,2,4,8,9,10
8	2,4,6	1,3,5,7,9,10
9	1,4,5	2,3,6,7,8,10
10	1,2,3	4,5,6,7,8,9

From the Table1, it is clear that $n_1 = 3 = 10 - n_2 - 1$, $n_2 = (3-1)\binom{3}{2} = 6$. For any two first associates, say, *j* and *j′ (j ≠ j′* =1, 2, …, 10), there is no common first associate to both *j* and *j′* i.e., P^1_{11}=0. Any two second associates, say, *j* and *j′ (j ≠ j′* =1, 2, …, 10*)*, have 1 common first associate to both *j* and *j′* i.e., P^2_{11} = 1. Thus, the parameters of the DLB Association Scheme are *v*=10, n_1=3, n_2=6, P^1_{11}=0, P^2_{11}=1.

III. CONSTRUCTION OF A NEW BLOCK DESIGN BASED ON DLB ASSOCIATION SCHEME

A result of the construction of PBIB design based on the DLB Association Scheme, starting from a BIBD is given as follows:

Theorem 3.1:

If there exists a BIBD, *D* $(v^*, b^*, r^*, k^*, \lambda^*)$ satisfying the three conditions for the DLB Association Scheme (given in section 2), then its dual design is a PBIB design based on DLB Association Scheme with parameters, $v= b^*$, $b= v^*$, $r= k^*$, $k= r^*$, $\lambda_1 = 1$, $\lambda_2 = 2$.

Proof: Consider a BIBD, *D* $(v^*, b^*, r^*, k^*, \lambda^*)$. As $k^* < v^*$ i.e., $r^* < b^*$, its dual is an Incomplete Block Design with parameters, $v= b^*$, $b= v^*$, $r= k^*$, $k= r^*$. It is given that the blocks of the given design intersect either one or two treatment(s) in common as the given design *D* satisfied the three conditions for DLB Association Schemes, taking any two blocks of the given design, say B_j and $B_{j'}$ *(j≠j')* intersecting one treatment in common. Then the two treatments corresponding to the two blocks of the parent design will occur exactly once in the dual of the given design. And further, since any two blocks B_j^* and $B_{j'}^*$ say *(j≠j')*, of the parent design intersecting two treatments in common, then the two treatments, j and j', corresponding to the two blocks of the parent design will occur exactly twice in the dual of the parent design. As any two treatments corresponding to the two blocks of the parent design intersecting i treatment(s) in common are i^{th} associates to one another *(i=1,2)*. Thus, we have, $\lambda_1 = 1, \lambda_2 = 2$. Hence the theorem.

Using the BIBD, Serial No. 4.4.1, Dey, A. (1986) [8], an example illustrating the Theorem 3.1 is given as follows:

Example 3.1**:** The BIBD has the parameters $v = 2t$, $b = 4t - 2$, $r = 2t - 1$, $k = t$, $\lambda = t - 1$. Taking $t = 3$, the parameters of the BIBD become $v^* = 6$, $b^* = 10$, $r^* = 5$, $k^* = 3$, $\lambda^* = 2$. The following is a plan of the BIBD with the parameters $v^* = 6$, $b^* = 10$, $r^* = 5$, $k^* = 3$, $\lambda^* = 2$, where 1, 2, …, 6 represent the treatments and parentheses indicate the blocks. B_1 = (1, 2, 3), B_2 = (1, 2, 4), B_3 = (1, 3, 5), B_4 = (1, 4, 6), B_5 = (1, 5, 6), B_6 = (2, 3, 6), B_7 = (2, 4, 5), B_8 = (2, 5, 6), B_9 = (3, 4, 6), B_{10} = (3, 4, 5). Then, by the Theorem 2.1, the number of treatments of the DLB association scheme is $v = 10$.

The table below describes the first and second associates of all 10 treatments.

Table 2: Association Table

Treatments	First associates	Second associates
1	4,5,7,8,9,10	2,3,6
2	3,5,6,8,9,10	1,4,7
3	2,4,6,7,8,9	1,5,10
4	1,3,6,7,8,10	2,5,9
5	1,2,6,7,9,10	3,4,8
6	2,3,4,5,7,10	1,8,9
7	1,3,4,5,6,9	2,8,10
8	1,2,3,4,9,10	5,6,7
9	1,2,3,5,7,8	4,6,10
10	1,2,4,5,6,8	3,7,9

From the Table 2, it is clear that $n_1 = 6 = 10 - n_2 - 1$, $n_2 = (2-1)\binom{3}{2} = 3$. For any two first associates, say, j and j' $(j \neq j' = 1, 2, \ldots, 10)$, there are 3 common first associate to both j and j' i.e., P_{11}^1=3. Any two second associates, say, l and l' $(l \neq l' = 1, 2, \ldots, 10)$, have 4 common first associate to both l and l' i.e., $P_{11}^2 = 4$. Thus, the parameters of the DLB Association Scheme are v=10, n_1=6, n_2=3, P_{11}^1=3, P_{11}^2=4.

The dual of the BIBD ($v^* = 6$, $b^* = 10$, $r^* = 5$, $k^* = 3$, $\lambda^* = 2$) is-

B_1:(1,2,3,4,5), B_2:(1,2,6,7,8), B_3:(1,3,6,9,10),
B_4:(2,4,7,9,10), B_5: (3,5,7,8,10), B_6: (4,5,6,8,9)

This incomplete block design is a PBIB design based on the DLB Association Scheme with parameters: v=10, b=6, r=3, k=5, $\lambda_1 = 1$, $\lambda_2 = 2$.

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