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Trend-free Circular Partially Balanced Incomplete Block design

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ABSTRACT- This paper constructs a series of linear trend-free circular (LTFC) partially balanced incomplete block (PBIB) designs for any block size k≥2. The proposed designs satisfy the requirements of PBIB designs. Theoretical proofs supporting the construction methods are presented, along with illustrative examples to demonstrate the proposed designs.

KEYWORDS- Linear Trend-Free Design; Circular Block Design; Partially Balanced Incomplete Block Design; Orthogonal Polynomial Trend.

I. INTRODUCTION

In experimental studies, heterogeneity among experimental units often affects the accuracy of treatment comparisons. Block designs are commonly used to control such variability when heterogeneity exists in a single direction. However, in many practical experiments, treatments are applied sequentially over time or space within blocks, which may introduce systematic trend effects. To overcome this problem, Bradley and Yeh [5] proposed the concept of trend-free block (TFB) in which treatment effects are orthogonal to polynomial trend effects, allowing unbiased estimation. Several researchers have contributed to the development of trend-free and trend-resistant incomplete block designs. Gupta et al. [7] constructed linear trend-free (LTF) partially balanced incomplete block (PBIB) designs of block size three, while Bhowmik et al. [1] developed linear trend-resistant PBIB designs for specific parameter settings. Motivated by these studies, the present paper proposes a series of linear trend-free circular partially balanced incomplete block (LTFCPBIB) designs for any block size k≥2. The proposed designs satisfy all PBIB design requirements and ensure resistance to linear trend effects. TFB design has been extensively studied in the literature by Bhowmik et al. [2]; Lal et al. [8]; Yeh and Bradley[10]. Additionally, foundational studies on PBIB designs provide structural support for developing such constructions, particularly the works of Bose and Nair[3]; Bose and Nair[4].

Let us consider v treatments be applied to plots arranged in b blocks, each of size k, $k \le v$. Assume a common polynomial trend of order p exists across the k positions, which can be represented using orthogonal polynomials $\phi_\alpha(l)$, $1 \le \alpha \le p$, on $l = 1, 2, \ldots, k$, where $\phi_\alpha(l)$ is a polynomial of degree α. Let the first block include the first k observations, the second block include the next k observations and so on. The polynomials $\phi_1(l)$, $\phi_2(l)$, …, $\phi_p(l)$ satisfy

$\sum_{l=1}^{k} \phi_\alpha(l) = 0$ and

$$\sum_{l=1}^{k} \phi_\alpha(l)\phi_{\alpha'}(l) = 1 \text{ if } \alpha = \alpha'$$
$$= 0 \text{ if } \alpha \neq \alpha' ;\ \alpha, \alpha' = 1, 2, \ldots, p.$$

The mathematical model for observation at position l in block j, $1 \le j \le b$, is

$$Y_{jl} = \mu + \sum_{i=1}^{v} d_{jl}^{i}\,\tau_i + \beta_j + \sum_{\alpha=1}^{p} \phi_\alpha \theta_\alpha + \varepsilon_{jl} \qquad (1)$$

where μ represent a general effect, $\tau_1, \tau_2, \ldots, \tau_v$ are the treatment effects, $\beta_1, \beta_2, \ldots, \beta_b$ are the block effects, θ_1, $\theta_2, \ldots, \theta_p$ are the trend effects.

Moreover, $d_{jl}^{i} = 1$, if treatment i is assigned to position l in block j; otherwise, it is 0. When $p = 1$ in (1), the design is referred to as a Linear Trend-Free Block design.

According to Lin and Dean [9], the polynomials $\phi_1(l)$ satisfy the condition $\phi_1(l) = -\phi_1(k - l + 1)$. In addition, $\phi_1((k+1)/2) = 0$, when k is odd.

Let a block design d will be represented by a $k \times b$ array using symbols 1, 2, …, v where rows correspond to blocks and columns to positions. Thus, if the entry in cell (j, l) of d is i, this indicates that treatment i is assigned to position l in block j under design d. Let $D(v, b, k)$ denote the set of all connected block designs involving v treatments arranged in b blocks with k positions each.

Let $d \in D(v, b, k)$ be a design and S_{dil} denote the number of times treatment i appears in position l. It has been shown by Chai and Majumdar [6], that a design is linear trend-free block (LTFB) design if

$$\sum_{l=1}^{k} S_{dil}\, \phi_1(l) = 0;\ i = 1, 2, \ldots, v \qquad (2)$$

where $\phi_1(l)$ is the orthogonal polynomials of degree 1; $l = 1, 2, \ldots, k$.

II. PROPOSITION

A proposition is presented to be used in the later part of the section, as follows.

Proposition 1 Developing an initial block $(\emptyset_1, \emptyset_2, \ldots, \emptyset_k)$; $\emptyset_i \epsilon$ module of v where none of $\emptyset's$ are equal to each other, a PBIB design with $v = b, r = k, \lambda_i;\ i = 1, 2, \ldots, \frac{v-1}{2}$ (for v odd) or $\frac{v}{2}$ (for v even) based on a Circular Association Scheme where any two elements at the distance i are the i^{th} associate with each other, can be constructed.

Proof: Consider any two treatments say θ and $\theta \pm i$; $i = 1, 2, 3, \ldots, \frac{v-1}{2}$ (for v odd) or $\frac{v}{2}$ (for v even). Then θ is the i^{th} associate of $\theta \pm i$ and vice versa.

For any treatment θ, all the i^{th} associates to θ are $\theta \pm i$. Therefore, the number of treatments that are the i^{th} associates of θ are $\theta \pm i$

i.e., $n_i = 2 \; \forall \; i = 1, 2, \ldots, \frac{v-1}{2}$ for v (odd),

$= 2, \forall \; i = 1, 2, 3, \ldots, [\frac{v}{2} - 1]$ for v (even),

$= 1$ for $i = \frac{v}{2}$.

Let us consider two cases of two mutually i^{th} associates viz.; (I) θ and $\theta + i$

Case I: θ and $\theta + i$ are two mutually i^{th} associates

Now, the j^{th} associate to θ are $\theta \pm j$ and the m^{th} associate to $\theta + i$ are $\theta + i \pm m$.

Therefore, $p^i_{jm} = \|\{\theta \pm j\} \cap \{\theta + i \pm m\}\|$

$= \|\{\theta + j, \theta - j\} \cap \{\theta + i + m, \theta + i - m\}\|$. (3)

For the possible values of p^i_{jm}, there are three possible subcases.

Subcase I

As i, j and m are known natural numbers for any θ

$\theta + j = \theta + i + m, \theta + j \neq \theta + i - m;$

$\theta + j = \theta + i - m, \theta + j \neq \theta + i + m;$

$\theta - j = \theta + i + m, \theta - j \neq \theta + i - m;$

$\theta - j = \theta + i - m, \theta - j \neq \theta + i + m;$

Subcase II

As i, j and m are known natural numbers for any θ

$\theta + j \neq \theta + i + m, \theta + j \neq \theta + i - m;$

$\theta - j \neq \theta + i - m, \theta - j \neq \theta + i + m;$

Subcase III

As i, j and m are known natural numbers for any θ

$\theta + j = \theta + i + m, \theta - j = \theta + i - m;$

$\theta + j = \theta + i - m, \theta - j = \theta + i + m.$

As i, j and m are fixed for particular p^i_{jm}, for varying values of θ exactly, only one of the subcases I, II, and III holds good. Hence, p^i_{jm} is unique as long as i, j and m have fixed values for a particular p^i_{jm}.

Thus, this relationship, i.e., θ is the i^{th} associate of $\theta + i$, satisfies the three conditions of the association scheme.

In the (3), the possible common elements between $\{\theta \pm j\}$ and $\{\theta + i \pm m\}$ can be in two different ways as follows:

(i) $\theta + j = \theta + i + m, \theta + j \neq \theta + i - m;$ or $\theta + j = \theta + i - m, \theta + j \neq \theta + i + m.$

(ii) $\theta - j = \theta + i + m, \theta - j \neq \theta + i - m;$ or $\theta - j = \theta + i - m, \theta - j \neq \theta + i + m.$

Then, $p^i_{jm} = 0$, if both (i) and (ii) are not satisfied,

$= 1$, if either (i) or (ii) is satisfied,

$= 2$, if both (i) and (ii) are satisfied.

Case II: θ and $\theta - i$ are two mutually i^{th} associates.

Now, the j^{th} associate to θ are $\theta \pm j$ and the m^{th} associate to $\theta - i$ are $\theta - i \pm m$.

Therefore, $p^i_{jm} = \|\{\theta \pm j\} \cap \{\theta - i \pm m\}\|$

$= \|\{\theta + j, \theta - j\} \cap \{\theta - i + m, \theta - i - m\}\|$. (4)

For the possible values of p^i_{jm}, there are three possible subcases

Subcase I:

As i, j and m are known natural numbers for any θ

$\theta + j = \theta - i + m, \theta + j \neq \theta - i - m;$

$\theta + j = \theta - i - m, \theta + j \neq \theta - i + m;$

$\theta - j = \theta - i + m, \theta - j \neq \theta - i - m;$

$\theta - j = \theta - i - m, \theta - j \neq \theta - i + m.$

Subcase II:

As i, j and m are known natural numbers for any θ

$\theta + j \neq \theta - i + m, \theta + j \neq \theta - i - m;$

$\theta - j \neq \theta - i - m, \theta - j \neq \theta - i + m.$

Subcase III:

As i, j and m are known natural numbers for any θ

$\theta + j = \theta - i + m, \theta - j = \theta - i - m;$

$\theta + j = \theta - i - m, \theta - j = \theta - i + m.$

As i, j and m are fixed for particular p^i_{jm}, for varying values of θ exactly, only one of the subcases I, II, and III holds good. Hence, p^i_{jm} is unique as long as i, j and m have fixed values for a particular p^i_{jm}.

Thus, this relationship, i.e., θ is the i^{th} associate of $\theta - i$, satisfies the three conditions of the Association Scheme.

In the (4), the possible common elements between $\{\theta \pm j\}$ and $\{\theta - i \pm m\}$ can be in two different ways as follows:

(i) $\theta + j = \theta - i + m, \theta + j \neq \theta - i - m;$ or $\theta + j = \theta - i - m, \theta + j \neq \theta - i + m.$

(ii) $\theta - j = \theta - i + m, \theta - j \neq \theta - i - m;$ or $\theta - j = \theta \mp i - m, \theta - j \neq \theta - i + m.$

Then, $p^i_{jm} = 0$, if both (i) and (ii) are not satisfied

$= 1$, if either (i) or (ii) is satisfied

$= 2$, if both (i) and (ii) are satisfied.

Remark: Such Circular Association Scheme may have $p^i_{jm} = 0, 1$ *or* 2 for different i, j and m.

III. CONSTRUCTION OF LTFCPBIB DESIGN

A. *For* $v = 2k - 1$

For $v = 2k - 1$, based on the Preposition 1, a series of LTFCPBIB designs, is constructed in the following theorem.

Theorem 3.1- Developing the initial block $\{1, 2, \ldots, k\}$ under the reduction mod($2k - 1$), the construction of LTFCPBIB design with the following parameters is always possible,

$v = b = 2k - 1, r = k, \lambda_1 = k - 1, \lambda_2 = k - 2, \ldots, \lambda_{k-1} = 1, n_1 = n_2 = n_3 = \cdots = n_{k-1} = 2.$

Proof: By developing $\{1,2, \ldots, k\}$ as the initial block of a design with mod($v = 2k - 1$), we get $2k - 1$ blocks. Therefore, $b = 2k - 1$.

By the nature of developing the initial block, every element in the different positions of the block gets replicated once.

As there are k elements (i.e., k positions) in the initial block and every element $i.e., 0, 1, 2, \ldots, (v - 1)$ appears once in each position. So, $r = k$.

For any treatment θ, all the j^{th} associates to θ are $\theta \pm j$; $\forall \; j = 1, 2, 3, \ldots, (k - 1)$.

Calculating all possible differences that arisen from the initial block $\{1,2, \ldots, k\}$, we get

Possible pairs	Number of pairs	their differences
$(1, 2), (2, 3), \ldots, (k - 1, k)$	$(k - 1)$	$\pm 1 \; i.e., 1, 2k - 2$
$(1, 3), (2, 4), \ldots, (k - 2, k)$	$(k - 2)$	$\pm 2 \; i.e., 2, 2k - 3$
$(1, 4), (2, 5), \ldots, (k - 3, k)$	$(k - 3)$	$\pm 3 \; i.e., 3, 2k - 4$
⋮	⋮	⋮
$(1, k - 1), (2, k)$,	2	$\pm(k - 2) \; i.e., (k - 2), (k + 1)$
$(1, k)$,	1	$\pm(k - 1) \; i.e., (k - 1), k.$

Then, the treatments at distance $\pm 1 \; i.e., 1, 2k - 2$ are the 1st associate;

the treatments at distance ± 2 $i.e., 2, 2k-3$ are the 2nd associate;
⋮
the treatments at distance $\pm k-2$ are $\pm(k-2)$ $i.e., (k-2), (k+1)$ are the $(k-2)^{th}$ associate;
the treatments at distance $\pm k-1$ are $i.e., (k-1), k$ are the $(k-1)^{th}$ associate.
So, $n_1 = n_2 = n_3 = \cdots = n_{k-1} = 2$.
Since θ and $\theta+1$ [θ and $\theta+(2k-2)$] are mutually 1st associate to one another, then they occur together in $(k-1)$ blocks. Therefore, $\lambda_1 = k-1$.
Similarly, $\lambda_2 = k-2, \lambda_3 = k-3, \ldots, \lambda_{k-1} = 1$.
Since every treatment occurs once in each position, then S_{dil}, the number of times that treatment i appears in position l is equal to 1 $\forall\, l = 1, 2, 3, \ldots, k$.
By, $\phi_1(1) = -\phi_1(k)$; $\phi_1(2) = -\phi_1(k-1)$ and so on.
Now, for $k =$ even

$$\sum_{l=1}^{k} S_{dil}\,\phi_1(l)$$
$$= \sum_{l=1}^{k} \phi_1(l)$$
$$= \phi_1(1) + \phi_1(2) + \cdots + \phi_1((k/2)-1) + \phi_1(k/2) + \phi_1((k/2)+1) + \cdots + \phi_1(k-1) + \phi_1(k)$$
$$= \phi_1(1) + \phi_1(2) + \cdots + \phi_1(k/2) - \phi_1(k/2) - \cdots - \phi_1(2) - \phi_1(1)$$
$$= 0.$$

Again, for $k =$ odd

$$\sum_{l=1}^{k} S_{dil}\,\phi_1(l)$$
$$= \sum_{l=1}^{k} \phi_1(l)$$
$$= \phi_1(1) + \phi_1(2) + \cdots + \phi_1(\frac{k+1}{2} - 1) + \phi_1\left(\frac{k+1}{2}\right) + \phi_1\left(\frac{k+1}{2} + 1\right) + \cdots + \phi_1(k-1) + \phi_1(k)$$
$$= \phi_1(1) + \phi_1(2) + \cdots + \phi_1\left(\frac{k+1}{2} - 1\right) + \phi_1\left(\frac{k+1}{2}\right) - \phi_1\left(\frac{k+1}{2} - 1\right) - \cdots - \phi_1(2) - \phi_1(1)$$
$$= 0.$$

Hence, proof of the theorem is complete.

The Theorem 3.1 is illustrated below through an example.

Example 3.1 The configuration of the LTFCPBIBD with the parameters $v = b = 7$, r = k = 4, $\lambda_1 = 3$, $\lambda_2 = 2, \lambda_3 = 1, n_1 = n_2 = n_3 = 2$ is as follows.
The first row in the configuration represents the orthogonal polynomial of degree one without normalization.

Block	Trend Component			
	-2	**-1**	**1**	**2**
B_1	1	2	3	4
B_2	2	3	4	5
B_3	3	4	5	6
B_4	4	5	6	0
B_5	5	6	0	1
B_6	6	0	1	2
B_7	0	1	2	3

Circular positions of 7 treatments in Circular Association scheme are shown in the below figure 1 and table 1.

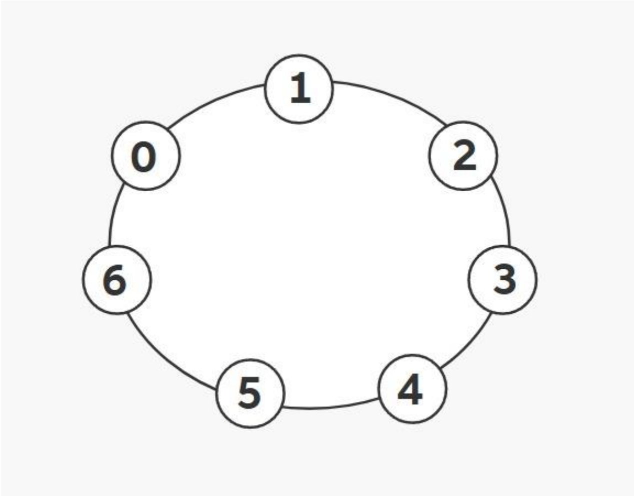

Figure 1: Circular Association Scheme for 7 treatments.

Table 1: The associates of each treatment for the above design

Treatments	1st associates	2nd associates	3rd associates
0	1, 6	2, 5	3, 4
1	2, 0	3, 6	4, 5
2	3, 1	4, 0	5, 6
3	2, 4	1, 5	6, 0
4	3, 5	2, 6	1, 0
5	4, 6	3, 0	1, 2
6	0, 5	1, 4	2, 3

Corollary 3.1 Developing the initial block $\{1, 2, \ldots, k\}$ under the reduction mod($2k$), the construction of LTFCPBIBD with the following parameters is always possible,

$$v = b = 2k, r = k, \lambda_1 = k-1, \lambda_2 = k-2, \ldots, \lambda_{k-1} = 1, \lambda_k = 0,$$
$$n_1 = n_2 = n_3 = \cdots = n_{k-1} = 2, n_k = 1.$$

Corollary 3.2 Developing the initial block $\{1,2, \ldots, k\}$ under the reduction mod$(2k+1)$, the construction of LTFCPBIBD with the following parameters is always possible,

$$v = b = 2k+1, r = k, \lambda_1 = k-1, \lambda_2 = k-2, \ldots, \lambda_{k-1} = 1, \lambda_k = 0,$$
$$n_1 = n_2 = n_3 = \cdots = n_{k-1} = n_k = 2.$$

Corollary 3.3 Developing the initial block $\{1,2, \ldots, k\}$ under the reduction mod$(2k-2)$, the construction of LTFCPBIBD with the following parameters is always possible,

$$v = b = 2k-2, r = k, \lambda_1 = k-1, \lambda_2 = k-2, \ldots, \lambda_{k-2} = 2, \lambda_{k-1} = 2,$$
$$n_1 = n_2 = n_3 = \cdots = n_{k-2} = 2, n_{k-1} = 1.$$

IV. CONCLUSION

This paper constructs a series of linear trend-free circular (LTFC) partially balanced incomplete block (PBIB) designs for any block size k≥2.

CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest.

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