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An Accurate Finite-Difference Method for 1-D Wave Problem Based on a Pre-Concerted Grid

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ABSTRACT

An accurate numerical method is proposed for the solution of 1-D initial-boundary value wave problem. It is of finite-difference type, where a pre-concerted grid is constructed using a specific time discretization. The ultimate goal is the minimizing of global error. Keeping of the stability-accuracy balance is the key element of the new proposal. It is particularly useful in case of large integration domains. The most important part of the method is implemented and tested in Matlab, and the results are compared with those of classic difference method.

Keywords

Initial-boundary value wave problem, Triangle of determinacy, Stability limit, Stability threshold, Non-uniform grid.

1. INTRODUCTION

There is an important class of engineering problems modeled by wave equations with known initial and boundary conditions. The explicit second order Finite Difference Method, (hereafter FDM), is often used to solve numerically such problems and in this case, time discretization is determined by the stability condition. But the global accuracy of FDM is low for problems involving large domains. The performance of computational processing, referred as efficiency, is also a big problem for large value of time variable. Accuracy and efficiency are the most urgent problems in wave problems. Many different differences schemes are proposed in literature to reduce the above cited undesirable effects. The locally adjusted time step strategies, like in [2] and [4], reduce considerably the computational cost involved. More sophisticated method, that combine variable grids with locally variable time-steps are used for big problems in seismic modeling. The staggered-grid technique, at beginning presented in [5], and developed further in [3], is four times as accurate as the conventional regular grid without increasing the amount of calculation. The accuracy and efficiency of finite-differences can be significantly improved combining staggered-grid with high-order finite-differences, but the algebra of these methods becomes intractable, and their implementation more difficult.

In this paper an 1-D wave problem is considered, so the accuracy matter prevails on that of efficiency. In next section we investigate the complex relation of global accuracy with the time and space discretization parameters of FDM. Based on findings in this section, a specific time discretization is proposed in section 3, which minimize the global error when applied for 1-D wave problem without boundary conditions. The non-uniform grid received by this discretization is used to solve the original initial-boundary value wave problem. Keeping of the stability-accuracy balance, when working near the stability limit of finite-differences, is the key element of new proposal. In case of non-homogenous equations the method operates in both sides of stability threshold parameter. The most important part of the method is tested in Matlab, and the results are compared with those of FDM.

2. GLOBAL ACCURACY OF FDM VERSUS ITS LOCAL ACCURACY

We will consider here the initial boundary-value problem for non-homogenous wave equation, with time-dependent boundary conditions, namely,

$$u_{tt} - u_{xx} = F(x, t) \tag{1}$$

with initial conditions.

$$u(x,0) = f(x),\ u_t(x,0) = g(x), \quad 0 \le x \le l, \tag{2}$$

and time-dependent boundary conditions

$$u(0,t) = p(t),\ u(l,t) = q(t), \quad 0 \le t \le T \tag{3}$$

We will consider also the associated initial value problem

$$u_{tt} - u_{xx} = F(x, t) \tag{1a}$$

with initial conditions.

$$u(x,0) = f(x),\ u_t(x,0) = g(x) \tag{2a}$$

It is difficult to obtain analytical solution for the problem (1-3), see e.x. [6]. Each problem requires specific treatment, moreover, analytical solutions in most cases are complicated, and sometimes inconvenient for practical use.

Denote by $D = [0\ l]x[0\ T]$ the physical space of variables x and t in (1-3). A grid G_0 is obtained by discretizing the interval $0 \le x \le l$ into m subintervals each of width h = l/m and interval $0 \le t \le T$ into N subintervals each of width k = T /N. FDM is given by the following equations

$u_{i,0} = f(x_i)$, i = 0,1,...,m+1;

$u_{0,j} = p(t_j)$, $u_{m+1,j} = q(t_j)$ (4)

for $j = 1,2,...,N$

$$u_{i,1} = (1-\lambda^2) f(x_i) + \frac{\lambda^2}{2}(f(x_{i-1}) + f(x_{i+1})) + kg(x_i) + \frac{h^2}{2} F(x_i, 0) \quad (5)$$

$$u_{i,j+1} = 2(1-\lambda^2)u_{i,j} + \lambda^2(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} + h^2 F(x_i, t_i) \quad (6)$$

for $j = 1,2,\text{L}\ N$, $i = 1,\text{L}\ ,m-1$, and $\lambda = k/h$.

The local error of method (4-6) is $O(h^2 + k^2)$, but it is numerical stable only if

$$\lambda = k/h \le 1 \quad (7)$$

The global accuracy of FDM has a complex relation with discretization parameters h and k. By the example below we will get some insight on this relation, using also the results of numerical experiments. Consider the initial-boundary value wave problem

$$u_{tt} - u_{xx} = 0$$

with initial conditions.

$$u(x,0) = \frac{4H}{l^2} x(l-x),\ u_t(x,0) = 0, \quad 0 \le x \le l,$$

and simple boundary conditions

$$u(0,t) = u(l,t) = 0, \quad 0 \le t \le T$$

The problem models the small vibrations of a string which is fixed at x = 0 and x = l. The initial displacement of the string is a parabolic segment with maximum displacement H small (In the demonstration below, we have taken H = 0.02, and l = 1.). The initial velocity is supposed zero. The exact solution of the problem, found by Fourier method, is

$$u(x,\ t) = \frac{32H}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin\frac{(2n+1)\pi}{L} x\ \cos\frac{(2n+1)\pi}{L} t$$

The FDM (4-6) is used in solving the problem for a range values of m and N, $N \le m$ ($k \le h$), as in Table 1. The proper selection of m and N is done for the best performance. The global accuracy is computed for each case as $\| u_{iN} - u(x_i,1)\|_2$ (L2-norm), i = 1, 2, ..., 6, and is presented in Table 1. In all experiments done with other examples, the corresponding tables look similar to Table 1.

Table 1: Global errors for the FDM solution at point (x0=0.5, t0=1), for different values of m and N

N / m	4	8	12	16	20	24
4	8.8459e-17	1.0054e-03	1.1471e-03	1.1913e-03	1.2109e-03	1.2214e-03
8	0	1.4706e-16	3.2656e-04	4.4412e-04	4.9249e-04	5.1678e-04
12	0	0	1.6419e-16	1.8289e-04	2.4743e-04	2.8337e-04
16	0	0	0	2.2232e-16	1.1713e-04	1.6145e-04
20	0	0	0	0	1.9645e-16	7.9701e-05
24	0	0	0	0	0	3.8529e-16

We could read through the results of Table 1:

a) For constant value of step size k (N - constant), the error decreases as h is decreases. This was expected. We can consider h as parameter of accuracy.

b) For constant value of step size h (m - constant), the error increases (although slightly) as k decreases. But the decreasing of k reduces the local error $O(h^2 + k^2)$, and also improves the stability according to (7) , so the increasing of global error was not expected.

c) For h = k, the global errors are extremely lower than the local errors would lead us to believe.

We will try to reason here the evident facts b) and c). In the Figure 1 below, OLM is the triangle of determinacy, R is an arbitrary point inside it, and PQR is the triangle of determinacy for point R. As the solution at point R is fully determined by the initial conditions in segment PQ, it is reasonable to assume that it would be meaningful for a numerical method to use only the approximations inside triangle PQR, for approximating in R. But if k > h, the integration method moves out the triangle PQR since in its first step. If k < h, we could never reach point R without using some approximations outside the triangle PQR.

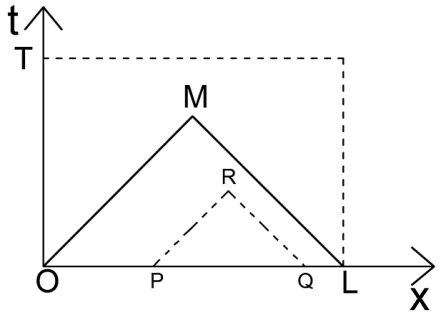

Figure 1: Triangle of determinacy of the solution

Roughly speaking, the number of these approximations is proportional to the difference h - k. What it happens inside the triangle OLM is spread outside it, meanwhile the boundary conditions begin to influence the solution. That explains the situation in case b). In special case c) when k = h, the approximation at point R does not use any previous approximations outside triangle PQR. The approximation is based only on the initial data that influence the solution at point R, and it uses all these data. This case seems meaningful and mathematically correct. However, the hint for the very accurate results of case c) is beyond this explanation. The local error of FDM (see e.x. [1]), can be written as

$$\frac{2}{4!}\left(k^2\frac{\partial^4 u(x_i,t_j)}{\partial t^4}-h^2\frac{\partial^4 u(x_i,t_j)}{\partial x^4}\right)+$$
$$+\frac{2}{6!}\left(k^4\frac{\partial^6 u(x_i,t_j)}{\partial t^6}-h^4\frac{\partial^6 u(x_i,t_j)}{\partial x^6}\right)+\mathrm{L}$$

By differentiating the homogeneous variant of equation (1) step by step, and after a few algebraic manipulations, we receive

$$\frac{\partial^{2n} u(x_i,t_j)}{\partial t^{2n}}=\frac{\partial^{2n} u(x_i,t_j)}{\partial x^{2n}},\ n=2,3,\mathrm{L} \tag{8}$$

Clearly the local error becomes zero for h = k, so the method (4-6) has no practically local error in case of homogenous equation, and the only errors are those due to the approximations of u_{i0} and u_{i1} in formulas (4-5). From the above discussion, the method (4-6), for h = k, seems to be rational for the solution of homogenous variant of problem (1)-(2)-(3), provided that the boundary T be a moderate number. But for large T, as it will be seen in the next section, the global accuracy of the method is decreased as T is increased. This is mainly because the theoretic qualities (8) do not hold numerically, they becomes approximations. We conclude this section emphasizing an important fact related to formula (7). For constant value of k, if we want to improve the stability of the method (4-6), we have to increase the value of h, this is to reduce the accuracy. If we want to increase the accuracy, we have to reduce h, but the best we could do is to take h = k. Stability versus accuracy: The effort to increase accuracy of many numerical methods is restrained by the demand to maintain their stability. So, for large value of T, the global accuracy of FDM has a complex relation with h, k, and λ. A good balance/compromise accuracy-stability, may be the key-factor of success.

A specific grid for the solution of initial-boundary value problem (1-3), different from the uniform grid of FDM, is proposed in the next section, based on the above reasoning.

3. THE CONSTRUCTION OF A GRID BASED ON GLOBAL ERROR MINIMIZATION

Consider initial wave problem (1a)-(2a). Let $M_0(x_0, t_0)$ be an arbitrary point, with $t_0 >> 0$, as in Figure 2. This is the point where the solution of problem (1a)-(2a) will be approximated. We can divide three cases to reach the point M_0, by applying the following difference method:

$$u_{i,0}=f(x_i),\quad \text{for } i=0,1,\ldots,2m+1 \tag{9}$$

$$u_{i,1}=(1-\lambda^2)f(x_i)+\frac{\lambda^2}{2}(f(x_{i-1})+f(x_{i+1}))+ kg(x_i)+\frac{h^2}{2}F(x_i,0) \tag{10}$$

$$u_{i,j+1}=2(1-\lambda^2)u_{i,j}+\lambda^2(u_{i-1,j}+u_{i+1,j})- -u_{i,j-1}+h^2F(x_i,t_i) \tag{11}$$

for $j=2,\mathrm{L}\ m,\ i=j+1,j+2,\mathrm{L}\ ,2m-j+1,$ and λ = k/h.

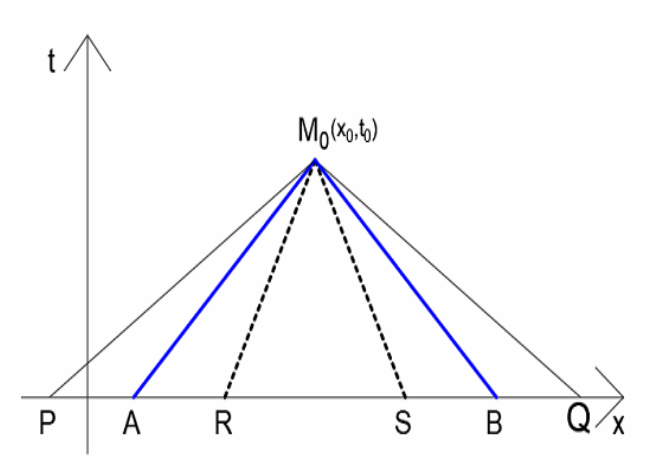

Figure 2: Three cases to reach the point M_0 by FDM

Case a) Using m steps of the processes (9-11), inside the triangle of determinacy ABM_0, with h = k = t0/m;

Case b) Using m steps of the processes (9-11), inside the triangle PQM_0, with h > k;

Case c) Using m steps of the processes (9-11), inside the triangle RSM_0, with h < k;

The step size $k = t_0/m$ is the same for the three cases. The cases a) and b) are both valid and they have their advantages and disadvantages. Case b) is "more" stable because $\lambda = k/h < 1$, but case a) is more accurate because step size h in case a) is less than that in case b). However the two cases are in a equilibrium balance and by routine numerical experiments it could be verified that they produce the same global accuracy at point M_0, for a wide specter of values of t_0. But the global accuracy of this difference method for large values of t_0 is poor. The case c) is more interesting. Clearly the process in case c), and in case of homogenous equation, is unstable, so it is not valid in this case. But as the stability analysis consider only homogeneous domains, it may be expected that under specific conditions, the case c) may be stable for non-homogeneous equation. It will be seen below. Let $M(x_0, t)$ be another arbitrary point with t in the neighborhood of t_0, as in Figure 3 or 4 below. Denote by ABM the triangle of determinacy for point M of Figure 3, and by PQM the triangle of determinacy at point M of Figure 4. Denote $t_1 = (t - t_0)$. There exists α, with |α| small, such that $t_1 = \alpha t_0$. It can be seen that α is positive in case of Figure 3 and negative in case of Figure 4. Let be n =1, 2, ..., 10. Denote $m = 2^{(n+1)}$. Take h = t/m. Denote $\epsilon = 2\alpha/(m(m-1))$. Consider the vector

$$kk = [h, h-\epsilon, h-2\epsilon, \dots, h-(m-1)\epsilon].$$

It can be verified that $|\epsilon| < h/(m-1)$, so the coordinates of vector kk are all positive decreasing numbers if $\alpha > 0$, and positive increasing if $\alpha < 0$. It can be verified also that:

$$h+(h-\epsilon)+(h-2\epsilon)+\dots+(h-(m-1)\epsilon) = t_0.$$

Denote $\lambda(j) = kk(j)/h$, $j = 1, 2, \dots, m$.

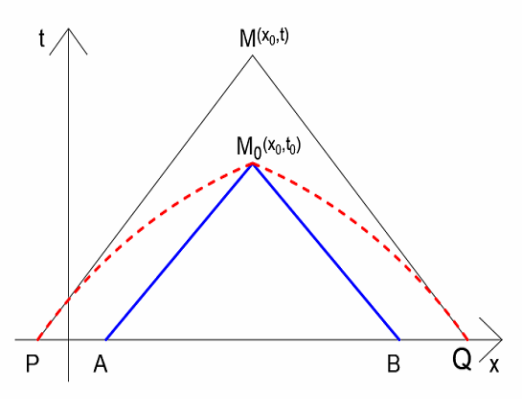

Figure 3: Approximating the solution of problem (1a)-(2a) at $M_0(x_0, t_0)$: Case $\alpha > 0$

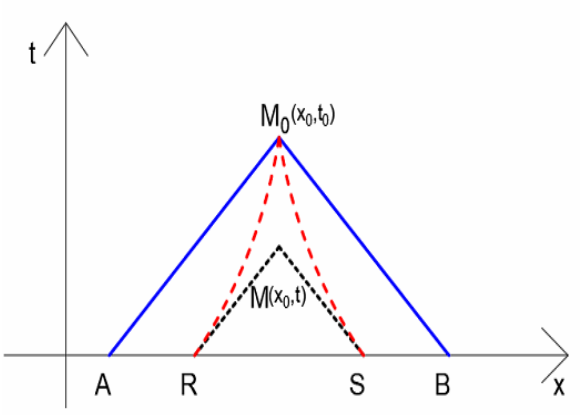

Figure 4: Approximating the solution of problem (1a)-(2a) at $M_0(x_0, t_0)$: Case $\alpha < 0$

Consider the following finite-difference method for approximating of the solution of problem (1a)-(2a) at point M_0

$$u_{i,0} = f(x_i),\ i=0,1,\dots,2m+1 \tag{12}$$

$$u_{i,1} = (1-\lambda(1)^2) f(x_i) + \frac{\lambda(1)^2}{2}(f(x_{i-1}) + f(x_{i+1})) + kg(x_i) + \frac{h^2}{2}F(x_i,0),\quad i = 2, 3, \dots, 2m \tag{13}$$

$$u_{i,2} = 2(1-\lambda(2)^2)u_{i,1} + \lambda(2)^2(u_{i-1,1} + u_{i+1,1}) - (u_{i,0} + \varepsilon f(x(i)),\quad i = 3, 4, \dots, 2m-1 \tag{14}$$

$$u_{i,j+1} = 2(1-\lambda(j)^2)u_{i,j} + \lambda(j)^2(u_{i-1,j} + u_{i+1,j}) - (u_{i,j-1} + \varepsilon(u_{i,j} - u_{i,j-1})/kk(j-1)) \tag{15}$$

$j = 2, 4, \dots, m$; $i = j+1, j+2, \dots, 2m-j+1$.

The equation (15) and its specific case (14) can be obtained by using the centered-difference quotient for the second partials, or simply writing the equation (11) in the context of Figure 5 below. This Figure corresponds to the case when α and therefore ϵ are positive. The analogous figure can be built for the case when α and therefore ϵ are negative. But the algebra related to deriving of formula (15), and also formula (14), includes both two cases.

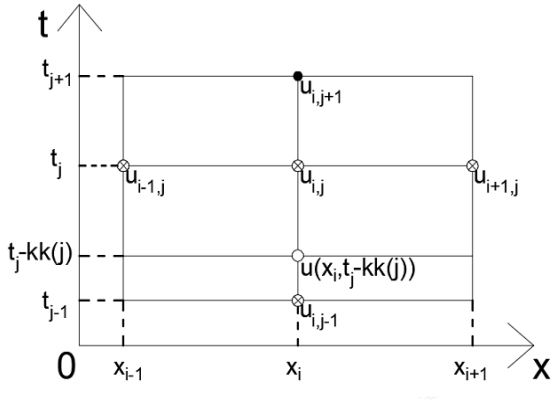

Figure 5: Scheme of nodes for computation at time t_{j+1}, case $\alpha>0$

In Figure 5, the symbol ● stands for the node where the solution is being approximated in the j^{th} step. The symbols ⊗ are used to denote the nodes where the solution was approximated in the previous steps. The symbol ○ stands for the point (xi, tj-kk(j)), which is the symmetric of the node-point (x_i, t_j+kk(j)), in relation to the line $t = t_j$. Also, according to the other notations in Figure 5, we can write: $kk(j) = t_{j+1}-t_j$, $kk(j-1) = t_j-t_{j-1}$, and $\lambda(j) = kk(j)/h$.

Writing the equation (11) for the points (x_i, t_{j+1}), (x_{i-1}, t_j), (x_i, t_j), (x_{i+1}, t_j), and (x_i, t_j- kk(j)) we obtain

$$u_{i,j+1} = 2(1-\lambda(j)^2)u_{i,j} + \lambda(j)^2(u_{i-1,j} + u_{i+1,j}) - u(x_i, t_j - kk(j)) \tag{16}$$

The value $u(x_i, t_j - kk(j))$ can be approximated by linear interpolation of nodes $u_{i,j-1}$ and $u_{i,j}$ as it follows:

$$u(x_i, t_j - kk(j)) = (u_{i,j-1} + (kk(j-1) - kk(j))(u_{i,j} - u_{i,j-1})/kk(j-1)) \tag{17}$$

Remembering that $kk(j-1) - kk(j) = \varepsilon$, and replacing the relation (17) into (16) we receive the equation (15).

The method (12-15) is presented symbolically by the red dotted trajectories in Figures 3 and 4. For value of α going to zero, the two red dotted trajectories, one being concave and the other convex, will converge to the same blue trajectory AM_0B, which symbolizes the classic FDM (9-11). This last can be obtained from (12-15) for $\alpha = 0$.

It can be seen that for a given $\alpha < 0$, method (12-15) uses a constant space step size $h(\alpha) = t/m > h$, and a variable time step size $kk(j) = h + (j-1)\epsilon > h$, so $\lambda(j) = kk(j)/h > 1$ for $j \geq 2$. That

means decreased accuracy (but increased efficiency) compared to FDM, and instability for the case of homogenous wave equation. It may look surprising to consider the case when α is negative. But as it will be seen in Example 2 below, this case, although looking as contrived, would results superior to the case when α is positive. For certain α we apply method (12-15) and find an approximation for $u(x_0, t_0)$. Denote this approximation as $u(\alpha)$. The global error can be computed as $g(\alpha) = |u(\alpha)-u(x_0, t_0)|$. Denote by α that value of α in a neighborhood of zero, for which the function $g(\alpha)$ takes its smallest value. Different strategies from global optimization can be used to find $\underline{\alpha}$. It can be proved that $\underline{\alpha}$ is depended from t_0, but not from x_0.

Now we return our attention at the initial-boundary value problem (1-3). We choose an arbitrary sufficiently refined small value for h. For t_0=T, we find as above the optimal value $\underline{\alpha}$(T). Clearly, based on h and $\underline{\alpha}$(T), a non-uniform grid G is created in the domain $D = [0\ l]x[0\ T]$. The method (4-6) can be easily adopted to solve the problem (1-3) in this pre-concerted grid. Other matters related to this adaptation will be discussed in next section.

4. MATLAB IMPLEMENTATION AND NUMERICAL RESULTS

The basic part of the Proposed Finite Difference Method described in section 3, hereafter referred as PFDM, is implemented in Matlab. Initial value problem (1a)-(2a) is solved for two cases of equation, homogeneous and non-homogenous. Approximations of the solution are provided at the points $M(x_0, t_0)$, for some typical values of t_0, selected for good performance. The corresponding optimal values $\underline{\alpha}(t_0)$ and global errors $g = g(\underline{\alpha}(t_0))$, are computed in each case. The problems are solved also by classic FDM in the same conditions as above. In order to compare different results, it makes sense to use the same space step size h in all integrations done. That's why the variable $m = 2^{(n+1)}$ and t_0 in Tables 2 and 3 below, change both progressively.

4.1 Homogenous Case, Example 1

Consider the initial value homogeneous wave problem

$$u_{tt} - u_{xx} = 0$$

with initial conditions

$$u(x,0) = x,\ u_t(x,0) = \sin x$$

The exact solution is

$$u = x + \sin x \sin t$$

The numerical results, as described above, for the arbitrary point x_0=100, are presented in Table 2. Global errors for FDM and PFDM are presented in the fourth row of this table.

Table 2: Global errors for the FDM and PFDM solutions at points (x0=100, t0), for different values of t_0

t_0	100	200	400	800	1600	3200
$\underline{\alpha}$	0.37656	0.10576	0.22992	0.025184	0.056184	0.0046953
n	6	7	8	9	10	11
FDM	2.8068e-02	4.8407e-02	4.7166e-02	4.9553e-02	4.4412e-02	5.3149e-02
PFDM	2.2737e-06	2.5508e-06	7.4332e-05	7.6287e-05	6.3759e-05	5.8329e-05

Clearly PFDM is superior on FDM for all the specter values of t_0 used. We can see that all $\underline{\alpha}$ are positive. This was expected: For $\alpha < 0$ PFDM is unstable because in this case, $\lambda(j) = kk(j)/h > 1$ for $j \geq 2$. From the Table 2 we can see also that the global errors of FDM are in their normal values, and cannot be compared with very accurate results of this method for small values of t, presented in Table 1.

4.2 Homogenous Case, Example 2

Consider the initial value non-homogeneous wave problem

$$u_{tt} - u_{xx} = 2$$

with initial conditions

$$u(x,0) = x^2,\ u_t(x,0) = \cos x$$

The exact solution is

$$u = x^2 + 2t^2 + \text{sintcosx}$$

The corresponding table of Table 2 for Example 2 is the following table.

Table 3: Global errors for the FDM and PFDM solutions at points (x0=100, t0), for different values of t_0

t_0	100	200	400	800	1600	3200
$\underline{\alpha}$	-3.021e-04	-2.618e-04	-1.314e-04	6.857e-05	-1.038e-05	1.831e-05
n	6	7	8	9	10	11
FDM	4.7798e-02	8.2435e-02	8.0322e-02	8.4386e-02	7.5634e-02	9.0503e-02
PFDM	1.3702e-04	1.4350e-04	5.2136e-05	1.2396e-04	6.7073e-04	1.4093e-04

The results of Table 3 may look surprising. In most of the cases the optimal values $\underline{\alpha}$ that minimize the global error are negative, meantime in case of homogenous equation all optimal α found were positive. In case of $\alpha < 0$, as it was mentioned before, $\lambda(j) = kk(j)/h > 1$ for $j \geq 2$. Stability analysis consider only homogeneous case. We see here that non-homogeneous case allows the use of less refined time discretizations, than homogeneous case does. According to the results of Tables 2 and 3, PFDM has at least two correct decimals more than FDM. The global space order of convergence and some other details of PFDM, its full implementation and further numerical results will be reported by the present authors in a coming paper.

5. CONCLUSIONS

An accurate numerical method of the finite-difference type is proposed and developed for the solution of 1-D wave equation. For homogenous equations it operates in the left side of stability limit $\lambda = 1$. For non-homogenous case it operates, in most of the cases, in the right side of this limit. The basic part of method is implemented in Matlab. According to experimental results, the new proposal gives two or three correct decimals more than FDM. The problem can be solved by economic effort for a considerably large interval of time, and the extra elapsed time compared to FDM, is fully justified by the accuracy provided.

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