International Journal of Innovative Research in Engineering & Management (IJIREM)
ISSN: 2350-0557, Volume-4, Issue-1, January-2017
DOI: 10.21276/ijirem.2017.4.1.3

Some Linear Transformations with MATLAB Applications, Especially On Infinite Integrals

D.A.GISMALLA
Faculty of Mathematics & Computer Sciences,
Gezira University,
Wad Medani, P.O.Box 20,
SUDAN

OHAMED H.A.ELHEBER
Faculty of Mathematics & Computer Sciences,
Gezira University,
Wad Medani, P.O.Box 20,
SUDAN

ABSTRACT

In this paper , We have considered , listed and described some linear transformations with their applications to sum infinite series accurately to a reasonable number of decimal places , e.g. at least to 10 decimal places of accuracy for most of the applications .We give attentions and especial consideration for evaluating an infinite integral section∮ 5 . This can be done by many linear transformations that can be found in the literature. However , here, We apply two fundamental Linear transformations called Euler's Transformations in section∮ 3. , Euler-Maclaurin Summation Formula in section∮ 4 and Levin's transform in .*section* ∮ 5.1.The Matlab program written for Euler's Transformation uses the diagonal elements for forward difference table as can be found in the Matlab File ForwadDifrenceTable.m in Fig.(6(b)) or obtained directly from

$$\Delta^{k}\mathbf{a_0} = \sum_{m=0}^{k}(-1)^{m}\binom{k}{m}\mathbf{a_{k-m}} \quad (1)$$

as in the Matlab File EulerTransform.m in Fig.(5) . This means the Matlab program written for Euler's Transformations uses slightly two different approaches and the reader should distinguish carefully between them. The most attractive in this paper the generalized linear transformations which are called Chebyshev Weights and Salzer's Means in section ∮ 2. For these two methods some fundamental background and the theory of generalized linear transformations are given to furnished to the reader how to follow the program to understand first the constraints on regularity or irregularity for these two method or similar ones .This means that the convergence of the method can be achieved only if these transformations are regular along any path. The path can be along zigzag ,column or a row. To be confident first that our program will be straight and accurately right ,We have used eqn.(3) and eqn.(4) to ensure regularity along any rows according to Toplitz limit theorem as in Theorem 3. Lastly in section ∮ 5 ,We consider the most robot method Levin's transform that have been pointed in many references such as [2] and [12]. Levin's transform have been analyzed and modified in many different ways to suit poorly slowly convergent series as in [2] .

KEY WORDS: Toeplitz limit. Chebyshev, Salzer and Levin's Transformations. Gaussian Quadtrue**.**

1. GENERALIZED TOEPLITZ TRANSFORMATIONS

The most famous result dealing with the regularity of linear transformations is the Toeplitz limit **Theorem 1** that concerns the convergence of transformations of B_s in Banach space B_s where the (n +1) th member of the transformed sequence is a weighted mean of the first n+1 members of the original sequence

$$\bar{\mathbf{s}}_{\mathbf{n}} = \sum_{k=0}^{n}\mu_{nk}\,\mathbf{s_k} \qquad (2)$$

The theory of this transformation is covered quite adequately in the existing literature (Knopp, 1947 , Hardy , 1956, Petersen , 1966, Peyerimhoff , 1969).

Here , We shall state an abstract version of the theorems that shall examine a convergence considerations for $\bar{s}_n$ *in Eqn.*(1) using Toeplitz Limit **Theorem 1** and its Toeplitz Regularity in **Theorem 2** as in [5]. Now , let ,T (s) = $\bar{s}$ is called a generalized Toeplitz transformation .

Theorem 1 (Toeplitz Limit Thearem).

The sum (1) converges , n≥ 0. and $\bar{s} \in B_c^2$ *iff*

(i) $\left\|\sum_{j=0}^{k}\mu_{nj}(s_j)\right\| \le M$ **for** $\|s_j\| \le 1$ **,** $j \ge 0$ **, and** $n, k \ge 0$**.**

(ii) $\sum_{k=0}^{\infty}\mu_{nk}(y)$ **,** $n \ge 0$ **, and** $\lim_{n\to\infty}\sum_{k=0}^{\infty}\mu_{nk}(y)$ **exist for** $y \in B^{I}$**,**

(iii) $\lim_{n\to\infty}\mu_{nk}(y)$ **exists for** $y \in B^{1}$**,** $k \ge 0$**.**

Theorem 2 (Toeplitz Regularity) .

T(s) = $\bar{s}$ is regular for B_c iff

(i) $\left\|\sum_{j=0}^{k} \mu_{nj}(s_j)\right\| \leq M \; for \|s_j\| \leq 1$

$, \mathbf{j} \geq \mathbf{0}, \mathbf{n}, \mathbf{k} \geq \mathbf{o};$

(ii) $\sum_{k=0}^{\infty} \mu_{nk}(y) = y + \sigma(1), n \geq 0, y \in B;$

(iii) $\mu_{nk}(y) = \sigma(1) in \quad n, k \geq 0, y \in B.$

1.1 Complex Toeplitz Methods

The only possible linear transformation is

$$\mu_{nk}(z) = \mu_{nk} z, \; \mu_{nk} \in B \quad (3)$$

In all practical situations $\mu_{nk} = 0$, k > n, and so the matrix

$$u = [\mu_{nk}] = \begin{bmatrix} \mu_{00} & 0 & 0 & \cdots \\ \mu_{10} & \mu_{11} & 0 & \cdots \\ \mu_{20} & \mu_{12} & \mu_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (4)$$

is lower triangular . If rows sum to 1,

$$\sum_{k=0}^{n} \mu_{nk} = 1, \quad (5)$$

then U, or the transformation defined by U,

T (S)= $\bar{s}$,

$$\bar{s}_n = \sum_{k=0}^{n} \mu_{nk} s_k, \quad n = 0, 1, 2, \ldots \quad (6)$$

is called a ***triangle*** .

We now restate the Toplitz limit theorem in a form suitable for U in Eqn.(6) in **Theorem 3**

Theorem 3 U is regular iff

(i) $\sum_{k=0}^{n} |\mu_{nk}| \leq M;$

(ii) $\sum_{k=0}^{n} \mu_{nk} = 1 + \sigma(1);$

(iii) $\mu_{nk} = \sigma(1), \; k \; fixed$

For the proof of these three theorems and the given in formations , the reader should see the pages (24-27) in [5]..

2. GENERALIZED LINEAR TRANSFORMATIONS

2.1 The Chebyshev Weights

What is probably the best of all the positive triangles results when $p_n(x)$ of the previous section is chosen to be the Chebyshev polynomial $T_n(x)$ of degree n . The efficiency of U in this case is a consequence of the extraordinary interpolatory properties of $T_n(x)$ that system of polynomials to play such an important role in numerical analysis and approximation theory.

$$T_n(x) = \cos n\theta,$$
$$\theta = \arccos x, \; n \geq 0. \quad (7)$$

so $T_0 = 1$

$$T_n(x) = \frac{1}{2} \begin{bmatrix} \left(x + \sqrt[i]{1 - x^2}\right)^n \\ + \left(x - \sqrt[i]{1 - x^2}\right)^n \end{bmatrix} \quad (8)$$

another useful representation is :

$$T_n(x) = {}_2F_1\left(\begin{matrix} n, -n \\ \frac{1}{2} \end{matrix} \middle| \frac{1-x}{2} \right)$$

letting $x = \frac{2\gamma}{a} + 1$ gives a positive triangle with entries

$$u_{nk} = \frac{(n)_k}{(\frac{1}{2})_k a^k} \; \frac{\binom{n}{k}}{\sigma_n(a)}$$

$$\sigma_n(a) = \frac{\left[\begin{matrix} \left(\sqrt{a+1} + 1\right)^{2n} + \\ \sqrt{a+1} - 1)^{2n} \end{matrix} \right]}{2a^n},$$

$k, n \geq 0.$

Usually one takes a = 1 . As an **example 1** of the power of U take a =1 in Eq. (5) and compute as in

Example 1

$$S_n = (\ln(2)_n = \sum_{k=0}^{n} \frac{(-1)^k}{k+1}$$

$\to \ln 2 = 0.69314718 \quad (9)$

The results are in Table 1 accurately to 14 D.Ps. and all the programs combines to do this are ChebyWeights.m and ChebyWeightSeries.m files in Fig.(1) and Fig.(2) respectively.

Table 1 shows error=
$\boldsymbol{log(2)} - \mathbf{0.693147180559954} = \mathbf{-8.659739e\text{-}15}$

n	su
1	1.000000000000000
2	0.666666666666667
3	0.686274509803922
4	0.693602693602694
5	0.693125361062970
6	0.693150956487264
7	0.693146851108180
8	0.693147230444046
9	0.693147174862919
10	0.693147181406835
11	0.693147180451170
12	0.693147180576273
13	0.693147180557710
14	0.693147180560286
15	0.693147180559898
16	0.693147180559954

2.2 Salzer Means

Salzer means are given by

$$\mu_{nk} = (-1)^{n+k} \frac{(\gamma + k)^n}{n!} \binom{n}{k}, \gamma > 0 \; (10)$$

and U transform is given by Eqn.(6) , similarly

ISSN: 2350-0557, Volume-4, Issue-1, January-2017
DOI: 10.21276/ijirem.2017.4.1.3

```
function [u]=ChebyWeights(n )

sigma=[(sqrt(2)+1)^(2*n)+(sqrt(2)-1)^(2*n)]/2;
u(1)=1/sigma;

for k=1:n
  u(k+1) =(n-k+1)*(n+k-1)*u(k)/[(0.5+k-1)*k ];
end
% disp(u);
```

```
for n=1:3
   [u]=ChebyWeights(n )
 end

 0.3333   0.6667

 0.0588   0.4706   0.4706

 0.0101   0.1818   0.4848   0.3232
```

Fig.(1) The Triangle Chebyshev Weights ChebyWeights.m with its Window Command

```
function [u]=SalzerMeans(n )

u(1)=(-1)^n /factorial(n);

for k=1:n
  u(k+1) =(-1)*(k+1)^n*(n-k+1)*u(k)/[k^(n+1) ];
end
end
      -----------------------------------------------
for n=1:3
   [u]=SalzerMeans(n )
  end

   -1     2

   0.5000   -4.0000    4.5000

  -0.1667    4.0000  -13.5000   10.6667
```

Fig.(3) The triangle Salzer Means for with its Window Command SalzerMeans.m

```
function [s,su]= ChebyWeightSeries(f ,a0,n)
 % In Matlab command window
 %  syms s;
 %  f=inline('(-1)^s/[s+1]');
 %  a0=1 ;
 %  n=3;
%****************
   su(1)=a0;
   for k=1:n
    [u]=ChebyWeights(k );
    [s]= DirectSum(f , a0 ,k);
   su(k+1)= s*u';
   end

 % The DirectSum.m flie for the series
 function [s]= DirectSum(f , a0 ,n)

 s(1)=a0;
   for  j=1:n
     a(j+1)=feval(f,j);
     s(j+1)=s(j)+a(j+1);
   end
 end
% The triangle ChebyWeights.m file
function [u]=ChebyWeights(n )

sigma=[(sqrt(2)+1)^(2*n)+(sqrt(2)-1)^(2*n)]/2;
u(1)=1/sigma;

for k=1:n
  u(k+1) =(n-k+1)*(n+k-1)*u(k)/[(0.5+k-1)*k ];
end

end
```

```
format long
syms s;
f=inline('(-1)^s/[s+1]');
a0=1 ;
n=15;
```

Fig.(2) The file ChebyWeightSeries.m with its Command Window for the series log(2) in Eqn.(1)
The output result in Table 1 correct to 14 D.P.

Example 2 Apply Salzer to compute

Sum of the series $\sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6}$ **(9)**

The results are in Table 2 accurately to 10 D.Ps.

The combined programs are in Fig.(3) shows **the** triangle Salzer Means for SalzerMeans.m with its Window Command while Fig.(4) shows the file SalzerMeansSeries.m with its Command Window for the series $\frac{\pi^2}{6}$ in Eqn.(9) .The output result in Table 2 **correct to 10 D.Ps**

3.EULER'S TRANSFORMATION OF SERIES

If $\sum_{k=0}^{\infty} (-1)^k a_k = a_0 - a_1 + a_2 - \cdots$ (11)

is a convergent series with sum s then

$s = \sum_{k=0}^{\infty} \frac{(-1)^k \Delta^k a_0}{2^{k+1}}$, where

$$\Delta^k a_0 = \sum_{m=0}^{k} (-1)^m \binom{k}{m} a_{k-m} \quad (12)$$

```
function [s,su]= SalzerMeansSeries(f ,a0,n)
 % In Matlab command window
 %  syms s;
 %  f=inline('1/[s+1]^2');
 %  a0=1 ;
 %  n=3;
%*****************
   su(1)=a0;
   for k=1:n
   [ u]=SalzerMeans(k );
    [s]= DirectSumSalzer(f , a0 ,k);
   su(k+1)= s*u';
   end
  disp([s(:) ,su(:)])
 % The DirectSumSalzer.m flie for the series
 function [s]= DirectSumSalzer(f , a0 ,n)
 % In Matlab command window
 % syms s;
 %  f=inline('1/[s+1]^2');
 %  a0=1 ;
 %  n=3;
 s(1)=a0;
 for  j=1:n
    a(j+1)=feval(f,j);
    s(j+1)=s(j)+a(j+1);
 end
 end

% The triangle SalzerMeans.m file
function [u]=SalzerMeans(n )

  u(1)=(-1)^n /factorial(n);

 for k=1:n
  u(k+1) =(-1)*(k+1)^n*(n-k+1)*u(k)/[k^(n+1) ];
 end
end
end
```

```
  In Matlab command window
  syms s;
 f=inline('1/[s+1]^2');
  a0=1 ;
  n=12;
```

Fig.(4) The file SalzerMeansSeries.m with its Command Window for the series $\frac{\pi^2}{6}$ in Eqn.(2) The output result in Table 2 correct to 10 D.P.

Example 3 Apply Euler's Transform to the series

$S_n = (\ln(2))_n = \sum_{k=0}^{n} \frac{(-1)^k}{k+1}$ (9)

The program Fig.(5) file EulerTransform.m with its CommandWindow for the series log(2) in Eqn.(9)

The output result correct to 11 D.P The sum of the series to 8 terms is 0.693147180540246 correct to 11 D.P. and with Table 3 to be followed.

Table 2 shows error= $\frac{\pi^2}{6}$ − 1.644934066775022= 7.3204331e-11

n	su
1	1.000000000000000
2	1.500000000000000
3	1.625000000000000
4	1.643518518518516
5	1.644965277777779
6	1.644951388888927
7	1.644935185185290
8	1.644933943418437
9	1.644934041170018
10	1.644934066247515
11	1.644934067166105
12	1.644934066862334
13	1.644934066775022

From Table 3 and Eqn.(12) we then obtain

$S = .634524 + \quad 24 + \frac{.111111}{2} - \frac{(-0.11111)}{2^2} + \frac{.002020}{2^3} - \frac{(-.000505)}{2^4} + \frac{.000156}{2^5}$

= .634524 +.055556 +.002778 + .000253 + .000032 + .000005 = .693148

(S = In 2= .6931472 to 7 D.P.)

Example 4 Evaluate the integral

$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ *to 4 D.P. using the Euler transform.*

$$\int_0^\infty \frac{\sin x}{x} dx = \sum_{k=0}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{\sin x}{x} dx$$

$$= \sum_{k=0}^{\infty} \int_0^{\pi} \frac{\sin(k\pi + t)}{k\pi + t} dt$$

$$= \sum_{k=0}^{\infty} (-1)^k \int_0^{\pi} \frac{\sin t}{k\pi + t} dt \quad (13)$$

Evaluating the integrals in the last sum by numerical integration we get Table (4) and from it the sum.

ISSN: 2350-0557, Volume-4, Issue-1, January-2017
DOI: 10.21276/ijirem.2017.4.1.3

The sum to k=3 is 1.49216. Applying the Euler transform to the remainder we obtain

$$\frac{1}{2}(.14180)-\frac{1}{2^2}(-.02587)+\frac{1}{2^3}(.00799)-\frac{1}{2^4}(.00321)+\frac{1}{2^5}(.00153)$$

$$=.07090+.00647+.00100+.00020+.00005=.07862$$

We obtain the value of the integral as 1.57078 as compared with 1.57080.

Table 3 The Forward Diagonal Difference Elements by Eqn.(12)

n	u_n	Δu_n	$\Delta^2 u_n$	$\Delta^3 u_n$	$\Delta^3 u_n$
9	.111111				
10	.100000	-11111			
11	.090909	- 9091	2020		
12	.083333	-7576	1515	-505	
13	.007692	-6410	1166	-349	156

Table 4 Gaussian Quadrature 5-points to evaluate $\int_0^{\pi}\frac{\sin t}{k\pi+t}$ using quassianTableInf.m Fig.(6) and the Diagonal elements by Eqn.(12)

$\int_0^{\pi}\frac{\sin t}{k\pi+t}dt \quad k=0,1,2,\dots,8$				
1.85194				
.43379				
.25661				
.18260	Δ	Δ^2	Δ^3	Δ^4
.14180				
.11593	-2587			
.09805	-1788	799		
.08495	-1310	478	-3211	
.07495	-1000	310	-168	156

In Example 4

$\int_0^{\pi}\frac{\sin t}{k\pi+t}$ using quassianTableInf.m Fig.(6)

and the Diagonal elements by Eqn.(12)

the value of integral $\int_0^{\infty}\frac{\sin x}{x}dx$ is computed accurately to 5 D.P. for k=8; in the command window in Fig.(6) . Howeverthe result is 1.570796318151585 when used only 5 point quadrature Gaussian Rule and so to get more accuracy one needs to use higher points in the rule.

This can be done by sight change in the file quassian Table In f.m ,where in the command window let k=18; to get the result accurately to 9 D.P. which is equal 1.570796318151585 and pi/2- 1.570796318151585 =8.643311e-09

```
function [sum]=EulerTransform(f ,n ,k )
sum1=0;
for j=0:n-1
   sum1=sum1 +feval(f ,0,j);
end
for j=0:k
   [d]=DiagonalForward(f ,j , n);
    diagonal(j+1)=d;
    Coef(j+1)=(-1)^j/2^(j+1);
end
sum=sum1+diagonal*Coef';

function [d]=DiagonalForward(f ,k , n)
% in th Command Window
%  n is the nth term in the series
% k is the kth term sum diagoanal diference
% k=12;
% n=8;
% syms s;
% syms n;
% f=inline(' (-1)^s/(n+1+s)');
sum=0;
   for  m=0:k
    s(m+1)=feval(f                              ,n,m)*(-
1)^k*factorial(k)/[factorial(m)*factorial(k-m)];
    sum=sum +s(m+1);
   end
   d=sum;
end
=====================================
 >> format long
>>  n=8;
>>  syms s;
>>  syms m;
>>  k=12;
>> [sum]=EulerTransform(f ,n ,k )
  sum = 0.693147180540246
  log(2)- 0.693147180540246= 1.96992422374e-11
==============================
```

Fig.(5) The file EulerTransform.m with its Command Window for the series log(2) in Eqn.(9). The output result correct to 11 D.P

The reader must observe that the computations for **Example 3**, **Example 4** and **Example 5** are cited form the reference Handbook of Mathematical functions [3] for which only their results are to 5 .D.P. We have been carried these to write the program for them that its goal to extend the accuracy and the examples are give so that they can be understand to be followed easily . The table 3 and Table 4 can be displayed easily from the file ForwadDifrenceTable.m or DiagonalForward.m ((but one require a little adjustment to employ an array variables instead of sum)). In ,**Example 5** below We will describe another important Euler's formula called **Euler − Maclaurin summation formula**

4.EULER-MALCAURIN SUMMATION FORMULA

$$\sum_{k=1}^{n-1} f_k = \int_0^n \int(k)dk - \frac{1}{2}\,[f(0)+f(n)] + \frac{1}{12}\left[f'(n) - f'^{(0)}\right] - \frac{1}{720}[f'''(n) - f'''(n)]$$

$$+\frac{1}{30240}[f^{(V)}(n) - f^{(V)}(0)]$$

$$-\frac{1}{1209600}\left[f^{(VII)}(n) - f^{(VII)}(0)\right] + \cdots (14)$$

Example 5.

Sum the series $\sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6}$ using the Euler − Maclaurin summation formula.

From Eqn.(3) we have for $n = \infty$,

$$\sum_{k=1}^{\infty} k^{-2} = \sum_{k=1}^{10} k^{-2} + \sum_{k=1}^{\infty} (k+10)^{-2}$$

$$= \sum_{k=1}^{10} k^{-2} + \int_0^{\infty} f(k)dk - \frac{1}{2}\, f_0 - \frac{1}{12}\, f'_0 + \frac{1}{720}\, f'''_0 - \cdots \quad (14)$$

$$\sum_{k=1}^{\infty} k^{-2} = 1.549767731 + .1 - .005$$

$$+.000166667 - .00000\,0333$$

$$= 1.64493\,4056$$

as compared with $\frac{\pi^2}{6} = 1.64493\,4067.$

The important about this formula in Eqn.(14) is that it combines both the methods for either evaluating an infinite series or an infinite integrals whenever the other one is known or it can be computed. While the reader must observe that there was a restriction for evaluating the first , the second , the third derivatives and so fourth ,etc., it can't be considered as a disappointed to the method for it can give a rough estimate if evaluating either one of them easily. This clearly seen here for the integral

$$\int_0^{\infty} (k+10)^{-2} dk \text{ in Eqn. (14) is easily computed}$$

Alternatively , the general Euler − Maclaurin summation formula is given

$$\sum_{k=a}^{\infty} f_k = \int_a^{\infty} f(x)dx + \frac{1}{2}\, f(a)$$

$$-\sum_{\ell=2}^{k} \frac{(-1)^{\ell}}{\ell!} f^{(\ell-1)}(a) B_{\ell}$$

$$-\frac{(-1)^k}{k!}\int_a^{\infty} f^{(k)}(x)\,\psi_k(x)dx \quad (15)$$

Where B_ℓ is Bernoulli numbers and

$\psi_k(x) = B_\ell$ *are the* Bernoulli polynomials

Despite the fact that in Example 5 the sum of the series $\sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6}$ is computed to 8 D.P. as We cited from [3] ,We give another Example 6 showing Euler − Maclaurin summation formula. *is not* recommended in general to evaluate an infinite integral

Example 6 Evaluate the incomplete gamma function $\Tau\left(-\alpha, \frac{-1}{z}\right)$ *when* $\alpha = 0$ *and* $z = -1$

i.e. evaluate the integral $\int_0^{\infty} \frac{e^{-x}}{1+x} dx$. Now apply Eqn.(14) as to get

$$\sum_{k=0}^{\infty} \frac{e^{-x}}{1+x} = \int_0^{\infty} \frac{e^{-x}}{1+x} dx + \frac{1}{2}\, f(0) + \frac{1}{12}[-2]$$

$$+\frac{1}{720}[16] + \frac{1}{30240}[-326]$$

$$-\frac{1}{1209600}[-13700] + \cdots (16)$$

Hence ,to evaluate $\int_0^{\infty} \frac{e^{-x}}{1+x} dx$ *in Eqn.* (16), *We* need to evaluate $\sum_{k=1}^{\infty} \frac{e^{-x}}{1+x}$ *by using Levin'straform as in Fig.* (7)

The result for this series is computed accurately to 0.283489078864866 to 12 D.P. From Eqn.(16) ,We get $\int_0^{\infty} \frac{e^{-x}}{1+x} dx =$ 0.6396 *which is incorrect* showing that in general Euler − Maclaurin summation formula must be avoided to evaluate an infinite integral.

5.EVALUATION ON INFINITE INTEGRALS

As ,We have seen ,above both Euler's Transformations and the rest of linear transformation s mentioned can be used to evaluate an infinite whenever it can expressed as a series .Further one of fundamentals series is Levin's Transform that discussed in [2] ,and [12]. Here , We applied to sum the series $\sum_{k=1}^{\infty} \frac{e^{-x}}{1+x}$ *by Fig.* (7) in the APPENDIX using Levin's Transform.

5.1 Levin's Transform

Levin's Transform is on e of the robot fundamental Transformations that compute most of the series expansions . In [2] , we have been shown that it can compute to more than 16 decimal places (D.P.) of accuracy using FORTRAN LANGUAGE with the FULL MACHINE ACCURACY. Unfortunately the program is not there but only a computational remarks in [12] expressing how to use the command to obtain the full machine accuracy whenever the FUNCTION LEVIN PRGRAM is called and expressed .

Fully when it is called .For such a reason the program is so long it can't followed to be practical.
However , when applied using FORMAT LONG with a MATALA PROGRAM it calculates similarly as in FORTAN LANGUAGE using full machine accuracy. Further , unfortunately ,Levin's transform breaks down to fail for very slowly convergent or divergent series as can be seen in [12]. Here ,We applied to compute $\int_0^\infty \frac{e^{-x}}{1+x}dx$ as in Example 7

Example 7 compute $\int_0^\infty \frac{e^{-x}}{1+x}dx$ using Levin$'s$ Transform to evaluate it

$$\int_0^\infty \frac{e^{-x}}{1+x}\,\mathrm{dx} = \sum_{\mathrm{k=0}}^{\infty}\int_{\mathrm{k}}^{(\mathrm{k+1})} \frac{e^{-x}}{1+x}\,dx \quad (17)$$

The result $\int_0^\infty \frac{e^{-x}}{1+x}$ dx = 0.5963473 accurate to 7 D.P. of accuracy as can be shown in Table 5 using only 10 terms while direct summation uses 20 terms .
Alternatively , apply partial integration the integral in Eqn.(17) can be expressed as in Eqn.(18)

$$\mathrm{I_x} = \int_x^\infty \frac{e^{-t}}{t}\,\mathrm{dt} = \frac{e^{-x}}{x}[1 - \frac{1!}{\mathrm{x}} + \frac{2!}{\mathrm{x}^2} - \frac{3!}{\mathrm{x}^3} + \cdots$$
$$+(-1)^{(\mathrm{n}-1)}\frac{(\mathrm{n}-1)!}{\mathrm{x}^{\mathrm{n}-1}}\ ] \ + R_n \quad (18)$$

$$\text{where } R_n = (-1)^n . n!. \int_x^\infty \frac{e^{-t}}{t}\,dt$$

Thus ,We have $|R_n| < e^{-x} n!/x^{n+1}$ and by definition

$$\frac{e^{-x}}{x}[1 - \frac{1!}{\mathrm{x}} + \frac{2!}{\mathrm{x}^2} - \frac{3!}{\mathrm{x}^3} +..+(-1)^{(\mathrm{n}-1)}\frac{(\mathrm{n}-1)!}{\mathrm{x}^{\mathrm{n}-1}}\ ..]$$

Hence, for large values of x the terms first decrease in absolute value but diverge when $n \to \infty$, $since\ x\ is\ fixed$. Now if x=10 and one uses direct summation to get

$$\mathrm{I_x} = \int_x^\infty \frac{e^{-t}}{t}\,\mathrm{dt} = e^{-10} * (0.0915633)$$

5.2Transform to Continued Fraction Technique

Since, as we have been shown in [6] series expansions cab be transformed to a continued fractions, we first expand the required integral

$$\mathrm{I_x} = \int_x^\infty \frac{e^{-t}}{1+t}\,\mathrm{dt} \quad x = 0 ,1 ,2 ,3,.. \ (19)$$

to $\mathrm{I_x} = \frac{e^{-x}}{1+x}[1 - \frac{1!}{1+\mathrm{x}} + \frac{2!}{(1+\mathrm{x})^2} - \frac{3!}{(1+\mathrm{x})^3} +$

$$....+(-1)^{(\mathrm{n}-1)}\frac{(\mathrm{n}-1)!}{(1+\mathrm{x})^{\mathrm{n}-1}}\ ..] \quad (20)$$

and then in general to C.F. as

$$\int_0^\infty \frac{\mathrm{e^{-t}}}{\mathrm{x+t}}\,\mathrm{dt} = \frac{1}{\mathrm{x}+1+}\frac{1}{\mathrm{x}+1+}\frac{1}{\mathrm{x}+1+}\frac{2}{\mathrm{x}+1+}\frac{2}{\mathrm{x}+1+}\frac{3}{\mathrm{x}+1+}\frac{3}{\mathrm{x}+}\ldots(21)$$

or as

$$e^{-x}\int_{\mathrm{x}}^\infty \frac{\mathrm{e^{-t}}}{\mathrm{t}}\,\mathrm{dt}$$
$$= \frac{1}{\mathrm{x}+1-}\frac{1}{\mathrm{x}+3-}\frac{4}{\mathrm{x}+5-}\frac{9}{\mathrm{x}+7-}\frac{16}{\mathrm{x}+9}\ldots \quad (22)$$

By evaluating Eqn.(21) and Eqn.(22) , We deduce that there are many so approaches can be used to evaluate infinite integrals or incomplete an infinite integral.

6. COMPUTATIONAL REMARKS

When writing a program for any one for the linear transformations Cheyshev or Salzer or similar transformations , one must check that the sum of absolute value elements of triangles u's in any row direction is equal to one before applying the series terms to find its sum.

When evaluating an infinite integral by expressing it as a series summations values of integrals ,then the ranges of these integrals must be very small as in Eqn.(17) or even be more smaller as [k*0.5 , (k+1)*0.5] for k=0 , 1, 2 , …. with Gaussian qaudrature rule one must use at least 10 points to get more accuracy rather than ,We have used 5 points .

Applying integration by parts the infinite integral

$$\int_0^\infty \frac{e^{-x}}{1+x}dx = \sum_{n=0}^{\infty}(-1)^n\, n! \qquad (23)$$

Eqn.(23) is an asymptotic divergent which can be sum with other alterative methods that may be ,We will be discussed them in the futures.


```
function InfiniteIntegration(a,b,n,k)
% n the number of points in Gaussain Quadratute
% k the number of remaining terms in
 %  Euler's Transform  % in the command window
% k=8;
% n=5;
% a=0;
% b=pi;
for k=0:k
   [Ie]=quassianTableInf(a,b,n,k);
   RIe(k+1)=Ie;
 end
 RIe'
 sum=0;
 for j=0:3
    sum=sum+(-1)^(j)*RIe(j+1);
 end
for j=5:k+1
    y(j-4)=RIe(j);
    Coef(j-4)= (-1)^(j-5)/2^(j-4);
 end
  [A] = ForwadDifrenceTable(y);
 % The diagonal elements of A multiplied by
 %     Coef=(-1)^k/2^k
   sum=sum+ Coef*A'
 % ************************************
 function [f]=myfun(x,k);
      f=  sin(x)/(k*pi+x);
 end
 % **************************************
  %   ForwadDifrenceTable.m  copied  in Fig.( )
   %*************************
%  Insert  file quassianTableInf.m in Fig.( )
end % end of file InfiniteIntegration.m
```

```
a=0; %  in the command window
b=pi;
n=5;
k=8;
format long
InfiniteIntegration(a,b,n,k)
ans =
   1.851937053223490
   0.433785479404046
   0.256610227718940
   0.182600578696064
   0.141803625692294
   0.115930889449606
   0.098051560386227
   0.084954236470784
   0.074945610486915
sum =
   1.570778548736482
```

The File InfiniteIntegration. m Fig. (6) combined with files quassianTableInf.m Fig.(6 (a)) and ForwadDifrenceTable.m Fig.(6(b)) to $\int_0^{\infty} \frac{\sin x}{x}$ **dx**

```
%*******************************************
function  [Ie]=quassianTableInf(a,b,n,k)
  %  Integration using  Guassian Quatrature rules
  % In Matlab command window
  % syms x;
  % f= inline(' sin(x)/(k*pi+x)');
  % a=0; b=pi ;
  % n=5 ; % Guassian-Table is given and n  equals 5
  % [Ie]=quassianTableInf(f,a,b,n,k )
 c=[1.0000000000         0.5555555556         0.3478548451
0.2369268850
      1.0000000000         0.8888888889         0.6521451549
0.4786286705
       0.0000000000         0.5555555556         0.6521451549
0.5688888889
       0.0000000000         0.0000000000         0.3478548451
0.4786286705
       0.0000000000         0.0000000000         0.0000000000
0.2369268850];
   x=[   0.5773502692        0.7745966692         0.8611363116
0.9061798459
      -0.5773502692         0.0000000000         0.3399810436
0.5384693101
       0.0000000000        -0.7745966692        -0.3399810436
0.0000000000
       0.0000000000      0.0000000000     -0.8611363116       -
0.5384693101
       0.0000000000      0.0000000000       0.0000000000        -
0.9061798459];
```

```
% ****************************************
  function [A] = ForwadDifrenceTable(y)
 % y = [y0,y1,...,yn-1] is the vector
 % of data values
  % F = n by n matrix of Forwad Differences.
  % A  Diagonal elements are the  Forwad
  %  Diffence Table
  %  F(1,1)
  %  F(2,1) F(2,2)
  %  F(3,1) F(3,2) F(3,3)
  % ...  ...   ...    ...     ...     ...
  %  F(n,1) F(n,2) F(n,3) F(n,4) ... ... F(n,n)
  n = length(y);
  F = zeros(n,n);
  A(1)= y(1);
% Construct first column using data values in y
   for i = 1:n
      F(i,1) = y(i);
   end
   % Construct Table diffrence table
    for j = 2:n % loop over column j
   % loop down column j from the diagonal
      for i = j:n
   % loop down column j from the diagonal
         F(i,j) = (F(i,j-1) - F(i-1,j-1)) ;
      end
    %  A is Diagonal elements are Forwad
    % Difference Table
      A(j) = F(j,j);
    end
    end     % Fig.(6(b))
%*********************************************
```


International Journal of Innovative Research in Engineering & Management (IJIREM)
ISSN: 2350-0557, Volume-4, Issue-1, January-2017
DOI: 10.21276/ijirem.2017.4.1.3

Table 5 Gaussian Quadrature 5-points to evaluate $I_k = \int_{k}^{(k+1)} \frac{e^{-x}}{1+x}$ k=0 ,1,2, ……20 using Direct Sum S_k & Levin's Sum $U_{(2k+2)}$

k	I_k	Direct Sum S_k	Levin's Sum
0	0.463421930906830	0.463421930906830	0.593821335725283
1	0.097456192252491	0.560878123159321	0.596341511720131
2	0.025195832234349	0.586073955393670	0.596347295689531
3	0.007151953938887	0.593225909332557	0.596347300371702
4	0.002142585452849	0.595368494785406	
5	0.000664893693863	0.596033388479269	Levin's Sum uses only
6	0.000211526114408	0.596244914593677	(2*k+2)=10
7	0.000068550361456	0.596313464955133	Terms . The result is
8	0.000022535603570	0.596336000558703	0.596347300371702
9	0.000007493402231	0.596343493960934	accurate to 7^{th} decimal
10	0.000002514932932	0.596346008893866	places
11	0.000000850608992	0.596346859502858	
12	0.000000289579408	0.596347149082266	
13	0.000000099135826	0.596347248218092	
14	0.000000034102998	0.596347282321089	
15	0.000000011781184	0.596347294102273	
16	0.000000004085096	0.596347298187369	
17	0.000000001421189	0.596347299608558	
18	0.000000000495890	0.596347300104448	
19	0.000000000173490	0.596347300277937	

6. ACKNOWLEDGEMENT

I would thank the International Centre of Theoretical Physics(ICTP) , ITALY for granting me a junior associate membership during 1988-1992
to send me both the references [4] and [5]due to an ARAB GRANT . However , I do BLAME the ARAB for they didn't follow it to be SEPENT all of it . In FACT , I have GOT LITTLE of it .

Second , I am gratefully of thanks to people , Government of SUDAN and University of Wales ,UNITED KINDOM to get my M.Sc. and Ph.D. and for which I have collect the precious reference [2].

REFERENCES

[1] Dafalla Awadalla Gismalla , Numerical Methods and Algorithms using Matlab Programming Language , LAMBERT Academic Publishing https://www.lap-publishing.com/site/how-to-publish/14

[2] D.A.Gismalla ,et al .Acceleration of Convergence of Series for Certain Multiple Integrals. International Journal of Computing Mathematics I. J.C.M, Vol. 24, pp 55-68, 1987.

[3] Abramowitz,M ,and Stegun ,I.A. 1964 ,Handbook of Mathematical Functions , Applied Mathematics Series , Volum 55 (Washington : National Bureau of Standards; reprinted 1968 by Dover Publication New York)

[4] Clark-Erik Frberg , Numerical Mathematics, Theory and Computer , Application. The Benjamin /Cummings Publishing Company, Inc , 1985

[5] Jet Wimp , Sequence Transformations and Their applications , Mathematics in Sciences and Engineering , Vol.154, 1981 ,Academic Press, London ,New York.

[6] D.A. Gismalla, Survey on Transformations for Infinite Series to Continued Fractions with MATLAB Program for Computing Some, International Journal of Innovative Research in Engineering & Management(IJIERM) ISSN:2350-0557,Volume 3 , Issue 3 , May 2016 http://www.ijirem.org/IJIREM/Arch.aspx?rep=12

[7] D.A.Gismalla On Evaluations Some Special Functions with Pade Approximant in Mupad Interface and Mathematica , International Journal Engineering Research & Management(IJERM) ISSN:2349-2058 , Volume 3 ,Issue 5 ,May 2016 https://www.ijerm.com/vol/Volume-03-Issue-05

[8] D.A.Gismalla & Mohamed H. A. ELHEBIR,Factorizations , Multiplications and Views on Parallel Computing. International Journal of Innovative Research in Engineering & Management (IJIREM) ISSN: 2350-0557, Volume-3, Issue-6, November-2016 http://www.ijirem.org/IJIREM/Arch.aspx?rep=15

[9] Petersen,G.M. (1966). "Regular Matrix Transformations". McGraw-Hill ,New York.

[10] Peyerimhoff.A. (1969)." Lecture Notes of Summability" . Lecture Notes In Mathematics

[11] Hardy ,G.H.(1956). " Divergent Series." Oxford Univ. Press , London and

[12] Methods for the Evaluations of N-dimensional Integrals, (a Thesis for the Ph.D. , Software was written using FORTRAN Language with Full Machine Accuracy), University of Wales, U.K. 1984 http://www.philadelphia.edu.jo/newlibrary/thesis-dissertation/406-mathematics/25864-4382

ISSN: 2350-0557, Volume-4, Issue-1, January-2017
DOI: 10.21276/ijirem.2017.4.1.3

APPENDIX

```
function   LevinTranfSum3(n ,a0)

% Integration using Levin Transform for Triple Integration on cubic.
  % This Technique is a series Technique for which the integral is expressed
  % as a series FIRST and then LEVIN is applied. The general term for the
  % series can be expressed as an inline function or a handle object with
  % the initial term submitted to the program in advanced to generate the
  % other terms with the number of terms n to be taken for the sum.
  % The function f in LevinTranfSum(f,n ,a0) is a ratio  to generate other terms .
  % In Matlab command window
  % a0= exp(-1)/2;
  % n=20;
   % LevinTranfSum(n ,a0)
   global UT  ;
     for k=1:n
        [S, UT]=LevinTransform(k ,a0);
         F(k)=UT;
     end
       disp(' The Sum of the Series having  2*n+2 terms')
       disp([ S']);
       disp(' The Sum of the Series using LEVIN TRANSFORM using 2* n+2  terms')
       disp([ F']);
  function  [ S , UT]=LevinTransform( k ,a0)
     a(1)=a0 ;
     S(1)= a(1) ;
     C(1)= 1;
   TotalSumDen(1)=1;
   TotalSumNum(1)=1;
  for j=1:2*k+1
       a(j+1)=  exp(-1)*(1+k)/(2+k)*a(j);
       S(j+1)=S(j)+a(j+1);
       C(j+1)=(2*k+2-j)*C(j)/(j);
       TotalSumDen(j+1)=TotalSumDen(j) + (-1)^j*C(j+1)*(j+1)^(2*k-1)*S(j+1)/a(j+1);
       TotalSumNum(j+1)=TotalSumNum(j) + (-1)^j*C(j+1)*(j+1)^(2*k-1)/a(j+1);
  end
       UT= TotalSumDen(2*k+2)/TotalSumNum(2*k+2);
```

```
format long
  n=20;
 a0=exp(-1)/2;
  LevinTranfSum3(n ,a0)
```

Fig.(7) is LevinTranfSum3.m to sum the series

$$\sum_{k=1}^{\infty}\frac{e^{-k}}{k+1} = 0.283489078864866 \text{ to 12 D.P.}$$

