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Summation Method for Some Special Series Exactly

D.A.Gismalla
Department of Mathematics & Computer Studies
Faculty of Science & Technology
Gezira University Wadi Medani,
Sudan

ABSTRACT

Efficient Numerical methods for series summation to high decimal places of accuracy can be found elsewhere. However, most of these methods can't sum many types of special series exactly because either of rounding's errors or these methods sometimes fail to compute slowly convergent series as in [1] . In this paper , We shall describe an approach that can be applied to sum some special types of series exactly whenever these special series can be emerged and suit our criterion method . Our method actually uses two approaches for expressing functions as series of Chebyshev polynomials approximation .The first approach is Taylor's Expansion Series where each mononomial x^n n=0,1,2,3,…. in Taylor's series is replaced and represented by its corresponding Chebyshev identity .The second approach is Cheybeshev Approximation Series for a particular function . Depending on this particular function, We compare the corresponding coefficients associated with $T_j(x)$ j=0,1,2,3,….. between the two series. Each coefficient in first approach emerge and generate an infinite series with its sum exactly equals the corresponding coefficient in the second expansion.

The particular function that We shall consider to be expressed first in Taylor's Method and second in Chebyshev Series to emerge the many infinite series is $\frac{1}{2}\ln[\frac{1+x}{1-x}]$ .The emerged infinite many series each with its exact sum are given by Eqn.(1)

$$\sum_{k=n}^{\infty}\frac{1}{2k-1}2^{-2k+2}C_{k+n-1}^{2k-1}=\frac{2}{2n-1}$$

$$for \quad n=1,2,3,4,...... \qquad (1)$$

where

C_{k+n-1}^{2k-1} *is Binomial Cofficient*

$$and \ C_{k+n-1}^{2k-1}=\frac{(2k-1)!}{(k-n)!(k+n-1)!}$$

Alternatively, Eqn.(1) can be rewritten as

$$\sum_{j=1}^{\infty}\frac{1}{2(j+n)-3}2^{-2(j+n)+4}C_{j+2n-2}^{2(j+n)-3}$$

$$=\frac{2}{2n-1} \quad for \quad n=1,2,3,4,......(2)$$

Eqn.(2) shows infinite formulae series that represents the reciprocal of odd numbers.

Keywords

Chebyshev polynomials, Taylor's Expansion, Binomial coefficient, chebyshev polynomials, Levin's Transform

1. THE VALIDITY OPINION FOR EXACT SUMMATION

Many infinite series having an exact summation can be found and evaluated it.

Here , for demonstration ,We list two infinite series ,each with its exact sum that can be evaluated easily . These series are in Eqn.(3) and Eqn.(4)

$$\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+2)}=1 \qquad (3)$$

We show the sum of the infinite series in Eqn.(3) is exactly equals one . Consider its partial sum S_n that can be expressed as

$$S_n=\frac{1}{1.2}+\frac{1}{2.3}+.......+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}$$

$$=(\frac{1}{1}-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+....+(\frac{1}{n}-\frac{1}{n+1})+(\frac{1}{n+1}-\frac{1}{n+2})$$

$$S_n=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+....+\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+1}-\frac{1}{n+2}$$

Observe that all the terms vanish with each other except the first and the last terms.
This implies that

$$S_n = 1 - \frac{1}{n+2}$$

Consequently , $s_n \to 1 \quad as \quad n \to \infty$

This shows that the infinite series in Eqn.(3) converges and its sum equals one , i.e.

$$S = \lim_{n\to\infty} S_n = \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = 1$$

Similarly, We can show , another infinite series in Eqn.(4) has its sum equals $\frac{\pi}{2}$ and it is given by

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k-1} = \frac{\pi}{2} \qquad (4)$$

This demonstrates that there are so many infinite series where each converges exactly to its sum without the need for seeking some numerical methods to to compute it. Such types of series can be useful for many applications**.**

2. THE ORTHOGONALITY AND THE FUNDAMENTAL IDENTITIES FOR CHEBYSHEV POLYNOMIALS

The Chebyshev polynomials of the first kind are defined through the identity

$$T_n(x) = \cos(n \ \cos^{-1} x) \qquad (5)$$

Now substitute x= $\cos(\theta) \quad to \quad get$

$$T_n(\cos(\theta)) = \cos(n\theta) \qquad (6)$$

Hence, the first few Chebyshev polynomials of the first kind are

$T_o(x) = 1$
$T_1(x) = x$
$T_2(x) = 2x^2 - 1$
$T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$

$T_5(x) = 16x^5 - 20x^3 + 5x$

$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$

The next identity expressed without proof is the expansion of Chebyshev polynomials $T_n(x)$ as successive powers in x , where n is a positive integer

$$T_n(x) = \sum_{j=0}^{r} \alpha_j x^{n-2j} \qquad (7)$$

where the coefficients α's are determined by

$\alpha_0 = 2^{n-1}$

and

$$\alpha_j = (-1)^j 2^{n-2j-1} C_{j-1}^{n-j-1} \frac{n}{j} \qquad (8)$$

for j = 1 (1) r and r = [n ⁄ 2] is the integer part of n ⁄ 2 .Where C_{j-1}^{n-j-1} is the corresponding Binomial coefficient. For an alternative expression of Eqn. (7), see [3].

The third well known identity that we have rewritten , is the explicit representation of x^n in terms of Chebyshev polynomial $T_j(x)$, j=0(1)n , for some positive integer n.

That is

$$x^n = {\sum_{j=0}^{n}}' c_j T_j(x) \qquad (9)$$

where $\quad c_j = 2^{-n+1} C_{(j+n)/2}^{n} \qquad (10)$

for j =1(1)n if n is odd & j =0(2)n if n is even . The prime dash in the summation means that coefficient of $T_0(x)$ in (9) should be halved . Further it can be shown that Chebyshev polynomials are orthogonal polynomials with respect to the weighting function $\frac{1}{\sqrt{1-x^2}}$

and satisfies Eqn.(11)

$$\int_{-1}^{+1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx = \int_0^{\pi} \cos(m\theta)\cos(n\theta) d\theta = \begin{cases} \frac{1}{2}\pi\delta_{nm} & for \ \ m \neq 0 \ , \ n \neq 0 \\ \pi \ \ for \ \ m = n = 0 \end{cases} \qquad (11)$$

Furthermore , Rodriguez representation identity is given by Eqn.(12).

$$T_n(x) = \frac{(-1)^n \sqrt{\pi} (1-x^2)^{\frac{1}{2}}}{2^2 (n - \frac{1}{2})!} *$$

$$\frac{d^n}{dx^n}[(1-x^2)^{n-1/2}] \qquad (12)$$

3. TAYLOR'S EXPANSION FOR $\frac{1}{2}\ln[\frac{1+x}{1-x}]$ WITH ITS MONONOMIAL x^n REPLACED BY CHEBYSHEV IDENTITY

Taylor's expansion as an infinite series for any function y(x) at the point $x_0 = 0$ is given by

$$y(x)=\sum_{k=0}^{\infty}\frac{x^k y^k(0)}{k!} \qquad (13)$$

Hence , Taylor's Expansion for $\frac{1}{2}\ln[\frac{1+x}{1-x}]$ at the point $x_0 = 0$ is given by

$$\frac{1}{2}\ln[\frac{1+x}{1-x}] = \sum_{k=1}^{\infty}\frac{x^{2k-1}}{2k-1} \qquad (14)$$

Observe that the power of each mononomial x^{2k-1} k=1 ,2, 3, …is odd .
This implies that ,We need to express each mononomial x^{2k-1} as Chebyshev identity by Eqn.(9) and Eqn.(10) . We must take n odd in Eqn.(10) , i.e. n=1 , 3, 5, 2n-1 . So , if , We replace each mononomial x^{2k-1} by its associate Chebyshev identity , Eqn.(14) becomes

$$\frac{1}{2}\ln[\frac{1+x}{1-x}] = \sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{1}{2k-1}2^{-2k+2}C_{k+n-1}^{2k-1}\right)T_{2n-1} \qquad (15)$$

Where C_{k+n-1}^{2k-1} is Binomial coefficient as in Eqn.(10)
Now , if We rewrite Eqn.(15) as Eqn.(16) below

$$\frac{1}{2}\ln[\frac{1+x}{1-x}] = \sum_{n=1}^{\infty}C_{2n-1}T_{2n-1} \qquad (16)$$

where the coefficient C_{2n-1} for n=1,2,3,4 ,…,is the sum for the infinite series inside the bracket in Eqn.(15) which must be determined . We get the many infinite series as

$$C_{2n-1} = \sum_{k=n}^{\infty}\frac{1}{2k-1}2^{-2k+2}C_{k+n-1}^{2k-1}$$

$$for \quad n = 1,2,3,.. \qquad (17)$$

The sum of each series in Eqn.(15) is exactly equals $\frac{2}{2n-1}$ $for\ n = 1,2,3,4,....$ as ,We will derive in the following section that $\frac{1}{2}\ln[\frac{1+x}{1-x}]$ can be expressed as an infinite Chebyshev series as in Eqn.(18)

$$\frac{1}{2}\ln[\frac{1+x}{1-x}] = \sum_{n=1}^{\infty}\frac{2}{2n-1}T_{2n-1} \qquad (18)$$

4. CHEBYSHEV EXPANSION SERIES FOR $\frac{1}{2}\ln[\frac{1+x}{1-x}]$

To develop the function f(x)=½ln[(1+x)/(1-x)] as a series of Chebyshev polynomials , We suppose that f(x) can be expressed as

$$\frac{1}{2}\ln[\frac{1+x}{1-x}] = \sum_{k=o}^{\infty}C_k T_k \qquad (19)$$

Now multiply both sides of Eqn.(19) with $T_n(x)$ and integrate with respect to the weight function $\frac{1}{\sqrt{1-x^2}}$ to get by using the orthogonality process as in Eqn.(11)

$$c_n = \frac{2}{\pi}\int_{-1}^{+1}\frac{\frac{1}{2}\ln[(1+x)/(1-x)]\ T_n(x)}{\sqrt{1-x^2}}dx \qquad (19)$$

The coefficient C_n =0 for n=0 ,2 ,4,….,2n and this can be seen from Eqn.(19) that the integrand in it is an odd function integrated on the interval [-1,1]. This implies that , We only seek to evaluate the odd coefficients C_n for n=1 ,3 ,5 ,…. , 2n-1 , …
Now substitute the Rodriguez representation Eqn.(12) into Eqn.(19)

$$T_n(x)=\frac{(-1)^n\sqrt{\pi}\sqrt{(1-x^2)}}{2^n(n-\frac{1}{2})!}\frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}} \quad (12)$$

and integrate by parts to get

$$c_n=\frac{2}{\pi}.\frac{(-1)^n\sqrt{\pi}}{2^n(n-\frac{1}{2})!}.\frac{1}{2}[0-$$

$$\int_{-1}^{+1}[\frac{1}{1+x}+\frac{1}{1-x}]\frac{d^{n-1}}{dx^{n-1}}(1-x^2)^{n-\frac{1}{2}}dx] \quad (20)$$

The reader must observed the limits can't be determined unless , We use the L'Hospital Rule in Eqn.(20) . Now , if We integrate by parts while differentiate (n-1)th times both the integrands in the bracket , We get

$$c_n=\frac{2}{\pi}.\frac{(-1)^n\sqrt{\pi}}{2^n(n-\frac{1}{2})!}.\frac{1}{2}[0+(n-1)!*\{$$

$$\int_{-1}^{+1}[(-1)^{2n-1}\frac{(1-x^2)^{n-\frac{1}{2}}}{(1+x)^{n-1}}$$

$$+(-1)^n\frac{(1-x^2)^{n-\frac{1}{2}}}{(1-x)^{n-1}}]dx]\} \quad (20)$$

Hence , the coefficients C_n for an odd numbers n=1 ,3 ,5 ,.. , 2n-1 , … can be rewritten as

$$c_n=\frac{2}{\pi}.\frac{\sqrt{\pi}}{2^n(n-\frac{1}{2})!}.\frac{1}{2}[(n-1)![I_1+I_2]] \quad (21)$$

Where the integrals in Eqn.(21) becomes

$$I_1=\int_{-1}^{+1}[\frac{(1-x)^{n-1}}{\sqrt{1-x^2}}-\frac{(1-x)^{n-1}x^2}{\sqrt{1-x^2}}]dx \quad (22)$$

and

$$I_2=\int_{-1}^{+1}[\frac{(1+x)^{n-1}}{\sqrt{1-x^2}}+\frac{(1+x)^{n-1}x^2}{\sqrt{1-x^2}}]dx \quad (23)$$

Observe that the integrals I_1 in Eqn.(22) and I_2 in Eqn.(23) contains two equals integrals given by

$$\int_{-1}^{+1}[\frac{(1+x)^{n-1}x^2}{\sqrt{1-x^2}}]dx=$$

$$\int_{-1}^{+1}[\frac{(1-x)^{n-1}x^2}{\sqrt{1-x^2}}]dx \quad (24)$$

which are both equal . Transforms one of the integrals in Eqn.(24) by substituting x=-y to get the equality .Further observes that these two integrals in Eqn.(14) contribute zero through the addition of I_1 & I_2 in Eqn.(21) .

Substitute x=cos θ into

$$I_1=\int_{-1}^{+1}[\frac{(1-x)^{n-1}}{\sqrt{1-x^2}}]dx$$

and x = cos θ into

$$I_2=\int_{-1}^{+1}[\frac{(1+x)^{n-1}}{\sqrt{1-x^2}}]dx$$

to get Wale's Formula

$$I_1=\int_0^{\pi/2}2^{n-1}\sin^{2(n-1)}\theta \; d\theta$$

$$=2^{n-1}\frac{2(n-1)!}{2^{2(n-1)}(n-1!)^2}\frac{\pi}{2}=I_2 \quad (22)$$

Now ,combine this in Eqn.(21) to get

$$c_n=\frac{2}{\pi}.\frac{\sqrt{\pi}}{2^n(n-\frac{1}{2})!}.\frac{1}{2}[\;(n-1)!*$$

$$2^n\frac{2(n-1)!}{2^{2(n-1)}(n-1!)^2}\frac{\pi}{2}] \quad (25)$$

Put

$$2(n-1)!=\frac{2\;(n-\frac{1}{2})!\;(n-1)!\;2^{2(n-1)}}{(2n-1)\;\sqrt{\pi}}$$

into Eqn.(25) to get

$$c_n=\frac{2}{2n-1} \quad n=1,2,3,..,\infty \quad (26)$$

Since the coefficients c_n's are zeros when n is even , then the coefficient in Eqn.(19) can be written as

$$c_{2k-1} = \frac{2}{2k-1} \quad k = 1,2,3,...,\infty \quad (26)$$

Hence the representation of the function f(x)=½ln[(1+x)/(1-x)] as Chebyshev Series can be expressed as

$$½\ln[(1+x)/(1-x)] = \sum_{k=1}^{\infty} \frac{2}{2k-1} T_{2k-1}(x) \quad (27)$$

Where $T_{2k-1}(x)$ is Chebyshev Polynomial of degree 2k-1

Now substitute the coefficients in Eqn.(17) into Eqn.(16) and compare with Eqn.(27) ,to get the required infinite many series with their exact sum as in Eqn.(1).

$$\sum_{k=n}^{\infty} \frac{1}{2k-1} 2^{-2k+2} C_{k+n-1}^{2k-1} = \frac{2}{2n-1}$$

$$for \quad n = 1,2,3,4,...... \qquad (1)$$

5. CONCLUSION

In a future work we shall investigate which best numerical methods are available to compute those infinite series in Eqn.(1) to a very high accuracy . It is well known that as We describe in [1] , Levin's Transform sum some types of infinite series to a very high accuracy using FORTRAN LANGUAGE . However , despite the fact Levin's Transform can be considered as one of those best numerical methods to sum infinite series , sometimes it has a drawback. It fails to sum some types of an infinite series and it can't evaluate convergent series exactly as in Eqn.(1) . In a future work , we will write software programs to compute the sum for infinite series to a high decimal places of accuracy . In particular ,We will apply the program to compute the series as type of Eqn.(1) .

REFERENCES

[1] D.A.Gismalla & A.M.Cohen Acceleration of Convergence of Series for Certain Multiple Integrals. I. J.C.M, Vol. 24, pp 55-68, 1987.

[2] Milton Abramowitz & Irene A_Stegun Handbook of Mathematical Functions Dover publications, Inc., NEW YORK

[3] Carl Erik Froberg Numerical Mathematics, Theory and computer Applications.The Benjimin / Cumming Publishing Company, Inc., 1985

[4] Chebyshev Approximations for cos(½ πx^4) and sin(½ πx^4) , 0≤x≤1 The Proceeding of International Conference on Computing I.C.C. Dec. 28th -29th .2010 , Newdlhi ,INDIA

