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A Note on Ideals and Regularity in Ternary Semigroups

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ABSTRACT
In this paper we have studied some properties of ideals in ternary semigroups.We have also studied some properties of regular ternary semigroups.We have presented some definitions and prepositions related to them.

Keywords
Ternary Semigroup, Left (Right, Lateral) Ideal, Regular Ternary Semigroup,

1. INTRODUCTION
J. Los[6] studied some properties of ternary semigroups and proved that every ternary semigroup can be embedded in a binary semigroup. D.H. Lehmer[5] gave the definition of a ternary semigroup.Banach showed by an example that a ternary semigroup does not necessarily reduce to a binary semigroup. F.M. Sioson[4] studied ideal theory in ternary semigroups and gave the definitions of ideals.The notion of regularity was introduced and studied by J.von Neumann [7] in 1936. F.M.Sioson [4] introduced the notion of regular ternary semigroups.

2. PRELIMINARIES
2.1 Definition
A ternary semigroup is a non-empty set S together with a ternary operation which has the property of association:

$$(abc)de = a(bcd)e = ab(cde)$$

for all a, b, c and d in S.

2.1.1 Example
The set $\{-i, 0, i\}$ is a ternary semigroup under the multiplication over complex number and is not a binary semigroup under this multiplication.

2.2 Definition
A non-empty set T of a ternary semigroup S is called a subsemigroup of S if $a \in T, b \in T$ and $c \in T$imply $abc \in T$.

2.3 Definition
A ternary semigroup S is called:
(i) left cancellative if $abx = aby$ implies$x = y$ for all a, b, x, y in S
(ii) right cancellative if $\mathrm{xab} = \mathrm{yab}$ implies$\mathrm{x} = \mathrm{y}$ for all $\mathrm{a, b, x, y}$ in S
(iii) lateral cancellative if $\mathrm{axb} = \mathrm{ayb}$ implies$\mathrm{x} = \mathrm{y}$ for all $\mathrm{a, b, x, y}$ in S
(iv)cancellative if S is left, right and lateral cancellative. □

2.4 Definition
An element e of a ternary semigroup S is called:
(i) left identity of S if $eea = a$for all a in S
(ii) right identity of S if $aee = a$for all a in S
(iii) lateral identity of S if $eae = a$ for all a in S
(iv) two sided identity of S if e is left and right identityof$S$
(v) identity of S if e is left, right and lateral identity of S.

2.5 Definition
An element z of ternary semigroup S is called a zero element of S if $zab = zza = zaz = azb = abz = azz = z$ for all a, b in S

2.5.1 Example
In the closed interval $I = [0, 1]$ we define $xyz = min(x, y, z)$ for all $x, y, z \in I$.
Let we show that 0 is a zero element and 1 is an identity of S. For all $x \in I$, $1xx = min\{1, x, x\} = x$, $x1x = min\{x, 1, x\} = x$, $xx1 = min\{x, x, 1\} = x$, $11x = min\{1,1, x\} = x$, $1x1 = min\{1, x, 1\} = x$dhe $1xx = min\{1, x, x\} = x$. Moreover, for all $c, y \in I$ we have $0xy = min\{0, x, y\} = 0$, $x0y = min\{x, 0, y\} = 0$, $xy0 = min\{x, y, 0\} = 0$, $x00 = min\{x, 0,0\} = 0$, $0x0 = min\{0, x, 0\} = 0$ and $x00 = min\{x, 0,0\} = 0$□

2.6 Preposition
Let X be any set and (o)a ternary operation in X such that $x\ o\ y\ o\ z = z$ for all x, y, zin X.Then, X is a ternary semigroup under the operation (o).

2.6.1 Proof
For all x, y, z, u, win X, $(x\ o\ y\ o\ z)\ o\ u\ o\ w = z\ o\ u\ o\ w = w$, $x\ o\ (y\ o\ z\ o\ u)\ o\ w = x\ o\ u\ o\ w = w$ and $x\ o\ y\ o\ (z\ \ o\ u\ o\ w) = x\ o\ y\ o\ w = w$□
A ternary semigroup $X(o)$ is called a right zero ternary semigroup in X.
In a similar way one may prove that:

2.7 Preposition
Let X be any set and$(*)$ a ternary operation in X such that $x * y * z = x$ for all x, y, zin X. Then, X is a ternary semigroup under the operation $(*)$□
A ternary semigroup$X(*)$ is called a left zero ternary semigroupin X.

2.8 Preposition
Let$S$ be a left zero ternary semigroup. Then $S$is right cancellative.

2.8.1 Proof
Let a, b, x, yin Ssuch that $xab = yab$. Since S is a left zero ternary semigroup we have $xab = x$and $yab = y$. Thus $x = y$.□
In a similar way one may prove that:

2.9 Preposition
Let$S$ be a right zero ternary semigroup. Then $S$is left cancellative.□

2.10 Preposition
An element e of a ternary semigroup S is called idempotent if
$eee = e$□
It is clear that a zero element and identity are idempotent elements. The converse is not true in general.
The sets $\{0\}$, $\{1\}$ and $\{e\}$, where e is an idempotent element, are ternary subsemigroups of S.

2.11 Preposition
Le e be an idempotent element of a left cancellative ternary semigroup S. Then e is a left identity of S.

2.11.1 Proof
For all $a \in S$, $eea = eeeeeea$ and since S is left cancellative we have $a = eeeea$. Also
$eee = e$imply$a = eea$□
In a similar way one may prove that:

2.12 Preposition
Le e be an idempotent element of a right cancellative ternary semigroup S. Then e is a right identity of S.□

2.13 Preposition
Let$S_1$and S_2be two ternary semigroups.The map$f: S_1 \to S_2$is calleda homomorphism if$f(abc) = f(a)f(b)f(c)$.□

3. IDEALS IN TERNARY SEMIGROUP
3.1 Definition
A non-empty subset A of a ternary semigroup S is called:
(i)left ideal of S if $SSA \subseteq A$
(ii)right ideal of S if $ASS \subseteq A$
(iii)lateral ideal of S if $SAS \subseteq A$
(iv)two sided ideal of S if A is a left and right ideal of S
(v)ideal of S if A is a left, right and lateral ideal of S. □

3.2 Definition
An ideal I of a ternary semigroup$S$with zero0is calledproper if $I \neq \{0\}$ dhe $I \neq S$

3.3 Definition
A ternary semigroup S is called:
(i) left simple if S has no proper left ideals
(ii) right simple if S has no proper right ideals
(iii) lateral simple if S has no proper lateral ideals
(iv) simple if S has no proper ideals.

It is clear that every left [right, lateral, two sided, ideal] ideal of a ternary semigroup$S$ is a ternary subsemigroup of S.

3.4 Preposition
Let S be a ternary semigroup. Then, SSa is a left ideal of S, for all a in S.

3.4.1 Proof
For all$x, y, s_1, s_2 \in S$we have$s_1 s_2 (xya) = (s_1 s_2 x)ya \in SSa$□
In a similar way one may prove that:

3.5 Preposition
Let S be a ternary semigroup. Then, aSS is a right ideal of S, for all a in S.

3.6 Preposition
A ternary semigroup$S$ is left simple if and only if $SSa = S$ for all $a$in S.

3.6.1 Proof
For all a in S, $SSa \subseteq S$. Preposition 2.4.implies that SSa is a left ideal of $S$and since $S$is left simple we have $SSa = S$.
Conversely, let Lbe a left ideal of S. For all$a \in L$we have $SSa = S$.This implies that for all$x \in S$, $x = bca$ with $b, c \in S$and since L is a left ideal of Swe have$x \in L$. Thus $S \subseteq L$. We have also that $L \subseteq S$. Hence $L = S$
In a similar way one may prove that:

3.7 Preposition
A ternary semigroup S is right simple if and only if $aSS = S$ for all a in S.

3.8 Definition
Let S be a ternary semigroup and a an element of S.
$(i)(a)_l = a \cup SSa$ is called the principal left ideal of S generated by a
$(ii)(a)_r = a \cup aSS$ is called the principal right ideal of S generated by a
$(iii)(a)_m = a \cup SaS \cup SSaSS$ is called the principal lateral ideal of S generated by a
$(iv)(a) = a \cup SSa \cup aSS \cup SaS \cup SSaSS$is called the principal ideal of S generated by a.

3.9 Preposition
Let S be a ternary semigroup. Then, for all $a \in S, (a)_r SS = aSS$ and $SS(a)_l = SSa$.

3.9.1 Proof
First let we show that $(a)_r SS = aSS$. It is clear that $aSS \subseteq (a)_r SS$, for all $a \in S$. Now let$x \in (a)_r s_1, s_2 \in S$.
If $x = a$ then $xs_1 s_2 = as_1 s_2 \in aSS$.
If $x \in aSS, x = as_3 s_4$ with $s_3, s_4 \in S$. Then, $xs_1 s_2 = (as_3 s_4)s_1 s_2 = as_3(s_4 s_1 s_2) \in aSS$. Thus$(a)_r SS = aSS$. Now let we show that $SS(a)_l = SSa$. It is clear that$SSa \subseteq SS(a)_l$. Let$s_1, s_2 \in S$and$x \in (a)_l$.
If $x = a, s_1 s_2 x = s_1 s_2 a \in SSa$.
If $x \in SSa, x = s_3 s_4 a$with$s_3, s_4 \in S$.
Then $s_1 s_2 x = s_1 s_2 (s_3 s_4 a) = (s_1 s_2 s_3) s_4 a \in SSa$.
Hence$SS(a)_l \subseteq SSa$.□

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3.10 Preposition

Let S be a ternary semigroup, L a left ideal of S and X a non-empty subset of S. Then, LXX is a left ideal of S.

3.10.1 Proof

Let $a \in L, x_1, x_2 \in X$ and $s_1, s_2 \in S$. Then, $s_1 s_2 (a x_1 x_2) = (s_1 s_2 a) x_1 x_2 \in LXX$ since $s_1 s_2 a \in L$ due to the fact that L is a left ideal of S

In a similar way one may prove that:

3.11 Preposition

Let S be a ternary semigroup , R a right ideal of S and X a non-empty subset of S. Then, XXR is a right ideal of S. □

3.12 Preposition

Let S be a ternary semigroup, L a left ideal of S, R a right ideal of S and M a lateral ideal of S. Then, LMR is a two sided ideal of S. Moreover $RML \subseteq R \cap M \cap L$.

3.12.1 Proof

Let $a \in L, b \in M, c \in R$ and $s_1, s_2 \in S$. Then, $s_1 s_2 (abc) = (s_1 s_2 a) bc \in LMR$ since $s_1 s_2 a \in L$ due to the fact that L is a left ideal of S. Thus LMR is a left ideal of S. We have also that $(abc) s_1 s_2 = ab(c s_1 s_2) \in LMR$ since $c s_1 s_2 \in R$ due to the fact that R is a right ideal of S. Thus LMR is a right ideal of S. Now let we show that $RML \subseteq R \cap M \cap L$. Let $a \in R, b \in M$ and $c \in L$. Then, $abc \in R$ since R is a right ideal of S. We have also that $abc \in M$ since M is a lateral ideal of S. On the other side $abc \in L$ since L is a left ideal of S. This implies that $abc \in R \cap M \cap L$

3.13 Preposition

Let A be a two sided ideal of a ternary semigroup S and B a two sided ideal of A such that $B^3 = B$. Then, B is a two sided ideal of S.

3.13.1 Proof

Since $B^3 = B$ we have that $SSB = SSBBB \subseteq SSAAB \subseteq AAB \subseteq B$. This implies that B is a left ideal of S. We have also that $BSS = BBBSS \subseteq BAASS \subseteq BAA \subseteq B$. This implies that B is a right ideal of S.

3.14 Definition

Let S be a ternary semigroup. An ideal I of S is called:

(i) strongly irreducible if $I_1 \cap I_2 \subseteq I$ implies $I_1 \subseteq I$ or $I_2 \subseteq I$

(ii) weakly irreducible if $I_1 \cap I_2 = I$ implies $I_1 = I$ or $I_2 = I$, for any two ideals I_1 and I_2 of S□

3.15 Definition

Let S be aternary semigroup. An ideal P of S is called:

(i) prime ideal of S if $I_1 I_2 I_3 \subseteq P$ implies $I_1 \subseteq P$ or $I_2 \subseteq P$ or $I_3 \subseteq P$ for any three ideals I_1, I_2, I_3 of S

(ii) completely prime ideal of S if $xyz \in P$ implies $x \in P$ or $y \in P$ or $z \in P$ for any three elements x, y, z of S□

3.16 Definition

Let S be a ternary semigroup. An ideal T of S is called:

(i) semiprimeif $III \subseteq T$ implies $I \subseteq T$ for any ideal I of S

(ii) completelysemiprimeif $xxx \in T$ implies $x \in T$ for any element x of S

4. REGULARITY IN TERNARY SEMIGROUPS

4.1 Definition

A ternary semigroup S is called regular if for all $a \in S$ exists $x, y \in S$ such that $a = axaya$.

It is clear that in a ternary semigroup S we have $SSS = S$.

4.2 Definition

An ideal I of a ternary semigroup S is called regular if $I \cup RML = R \cap M \cap L$ for any right ideal $R \supseteq I$, lateral ideal $M \supseteq I$, left ideal $L \supseteq I$.

4.3 Preposition

Every regular and strongly irreducible ideal is prime.

4.3.1 Proof

Let S be a ternary semigroup and I a regular and strongly irreducible ideal of S. Let $I_1 I_2 I_3 \subseteq I$. Then , $I_1 I_2 I_3 \cup I = I$. Also, since I is a regular ideal we have $I_1 I_2 I_3 \cup I = I_1 \cap I_2 \cap I_3 \subseteq I$ and since I is strongly irreducible we have $I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$. When ce I is prime.

4.4 Preposition

Every regular ideal is semiprime.

4.4.1 Proof

Let S be a ternary semigroup and I a regular ideal of S. Let JJJ⊆I. Then, JJJ ∪ I = I. Since I is a regular ideal we have JJJ ∪ I = J ∩ J ∩ J = J⊆I . Hence Iis semiprime.

4.5 Lemma

Let $\Phi : S \to T$ be a one-to-one homomorphism of a ternary semigroup S to a ternary semigroup T. Then, imΦ is regular. If f is an idempotent element of imΦ, then exist an idempotent element e of S such that $\Phi(e) = f$.

4.5.1 Proof

For all $z \in$ imΦ, $z = \Phi(a)$ with $a \in S$. Since S is regular, exists $x, y \in S$ such that $a = axaya$. Then, $\Phi(a) = \Phi(a)\Phi(x)\Phi(a)\Phi(y)\Phi(a)$ since Φ is homomorphism.Thus $z = zuzvz$ with $u = \Phi(x)$ and $v = \Phi(y)$. This implies that imΦ is regular. Now let f be an idempotent element of imΦ. Then, $fff = f$ and exist $e \in S$ such that $\Phi(e) = f$. Thus $f = \Phi(e)\Phi(e)\Phi(e) = \Phi(eee)$ and since Φ is one-to-one we have $eee = e$.

4.6 Preposition

Let S a regular ternary semigroup, R a right ideal of S, L a left ideal of S and M a lateral ideal of S. Then, $RML = R \cap M \cap L$.

4.6.1 Proof

Preposition 2.12.implies $RML \subseteq R \cap M \cap L$. Let $a \in R \cap M \cap L$. Since S is regular exists $x, y \in S$ such that $a = axaya$. Since $a \in M$ and M is a lateral ideal of S we have $xay \in M$. We have also that $a \in R$ and $a \in L$. Hence $a \in RML$.

5. CONCLUSION

Many concepts in binary semigroups extend similarly to ternary semigroups. In this section we presented some properties of ideals in ternary semigroups and regular ternary semigroups. All of these help us to find out the structure of a ternary semigroup.

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